Topics in Statistical Physics and Probability Theory

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WebPage on Ron's website. Grade will be homework. Book: Friedli & Velenik. Lecture notes on webpage.

What is statistical physics: Let us think of a magnet, if we look at it globally in the macroscopic level we use thermodynamics. We can measure macroscopicaly its U-Energy, V-Volume, M-total magnetization, N-Number of particles, P-Pressure, T-Temprature and μ-chemical potential. Ideal gas equations: Relations amongst the quantities. For exmple in a gas, N, V determine everything else.

1 Statistical Physics - Physical Intro

Microscopic constituents. For exmple we can think of the magnet as its atoms, and the interactions between them.

1.1 Phase Transitions

Magnet: on the macroscopic level, we apply a magnetic field on the magnet in a specific direction, to enforce a certain magnetization, in the direction of the field. Let us say we applied a magnetic field of size h. Pierre curie asked what will happen if we do the same experiment with a reduced size of the magnetic field. One possibility is that the magnet is a parmagent and then the graph looks like arctangens. Another option is that the magnet is a ferromagnet for which there is a jump in 0 , between a positive value and a negative value, with the value of zero a positive number. When the value of 0 is not 0 we say that the material exibits spontaneous magnetization.

Tempratue: Pierre observed that this question is related to the temprature. In high tempratures the material is a paramanget, andin low tempratures the material is a ferromagnet. Interestingly there is a **Transition point** T_c called the "Curie Temperature".

We have two kinds of phase transitions. A **Continuous Phase Transition** and a **Discontinuous Phase Transition**.

2 Statistical Physics Models

In every model we have an alphabet $\mathfrak{A}$ (usually either S^n or $\mathbb{R}$), a graph $G = (\Lambda, E(G))$ and a configuration $\omega : \Lambda \to \mathfrak{A}$.

2.1 Gibbs Distribution

The Hamiltonian H is a function from $\mathfrak{A}^\Lambda$ to $\mathbb{R}$, the energy of a configuration. We define β to be the inverse temperature ($\frac{1}{T}$) we will assume it to be positive. We have the gibbs boltzman distribution as the probability measure $P_{\beta,H} = \frac{1}{Z_{\beta,H}} e^{-\beta H(\omega)}$ with $Z_{\beta,H}$ the normalization constant (sometimes called the partition function). In this distribution lower energy configurations are favored. How much are they favored? at high β they have larger probability, and at low β the distribution is almost uniform. Observing the model in equilibrium is when we study the model with this distribution.

2.2 Ising Model

$\mathfrak{A} = S^0$ and the graph is a finite box in a lattice. We are given $h \in \mathbb{R}$, the external magnetic field. We define $H(\omega) = -\big(\sum\limits_{i,j\in\Lambda, i\sim j} \omega_i\omega_j + h\sum\limits_{i\in\Lambda}\omega_i\big)$. Then the probability measure is $\frac{1}{Z_\beta}e^{\beta\big(\sum\limits_{i,j\in\Lambda,i\sim j}\omega_i\omega_j + h\sum\limits_{i\in\Lambda}\omega_i\big)}$. We define the total magnetization: $M = \sum\limits_i \omega_i$ and the magnetization density: $\frac{M}{|\Lambda|}$. The macroscopic quantities are expectation values, for expmle $\mathbb{E}[\frac{M}{|\Lambda|}]$ and $\mathbb{E}[\frac{|M|}{|\Lambda|}]$.

2.2.1 The History of The Ising Model

It was first introduced by Wilhelm Lenz in 1920. He gave it to his P.h.D student Ernst Ising 1925. He concludes that there is no phase transition in one dimension $\Lambda = \{-n, ..., n\}$ and he thinks that there is no phase transition in any dimension, and therefore it is not a good model. In 1936 Rudulph Peierls argues mathematically that there is a phase transition in $d \geq 2$. Lars Onsager solved the 2d Ising model in 1944 with $h = 0$, i.e. he calculated a formula for the distribution, and then it really took of. What Onsager did is to take $\Lambda = \{-n, ..., n\}^2$ and on this graph he calculated the expectation value $\mathbb{E}[\frac{|M_n|}{|\Lambda|}] \overset{n\to\infty}{\to} m(\beta)$ for $h = 0$. Writing $p := 1 - e^{-2\beta}$ then $m(\beta) = m(p) = 0$ when $p < p_c$ and $(1 - (\frac{2(1-p)}{p(2-p)})^4)^{\frac{1}{8}}$ for $p > p_c$ with $p_c = \frac{\sqrt{2}}{1+\sqrt{2}}$. The phase transition is continuous. What happens in 1d? For any given n we need $\beta > \beta(n)$ to see almost every spin 1 or almost every spin -1, however $\beta(n) \overset{n\to\infty}{\to} \infty$.

3 Topics

We will start with the Ising model on the complete graph - Curie Weiss Model. Then we will pass to the $\mathbb{Z}^d$ version - high temperature, low temperature, critical?.

After theIsing model we will speak of the Spin O(n) Models: the alphabet is S^1 (XY model) or S^2 (Fleisenberg Model) with $H(\omega) = -\sum\limits_{i\sim j} < \omega_i, \omega_j >$, we will show that there is no phase transition in $d = 2$ and there is a phase transition $d \geq 3$.

We may discuss the Loop O(n) model, a graphical representation of spin O(n) models and connections to SLE.

After that we will talk about Random Surfaces $\mathfrak{A} = \mathbb{R}$, with $H(\omega) = \sum\limits_{i\sim j}(\omega_i - \omega_j)^2$ the discrete Gaussian Free Field.