Topics in Discrepancy (Barak Weiss)

Site: www.math.tau.ac.il/~barakw/discrepancy

There will be an (updating) problem sheet.

#1 References: Matousek – Geometric Discrepancy

Beck and Chen – Irregularities of Distribution

W.M. Schmidt – Lectures on Irregularities of Distribution

??? and Tichy – Sequences, Discrepancies and applications

We will survey 2 topics that lead to the same question:

① Numerical Integration

② Effective Ergodic Theorems (Diophantine Approximations)

① Let $\mathcal{U}=[0,1]$, $\mathcal{U}^d=[0,1]^d$. We have $f:\mathcal{U}\to\mathbb{R}$ which is RI – Riemann Integrable.

We want to know $\int_0^1 f(x)dx$. If $f(x)=F'(x)$ then $\int_0^1 f(x)dx=F(1)-F(0)$.

We approximate $\int_0^1 f(x)dx$ by $\frac{1}{N}\sum_{i=0}^{N-1} f(x_i)$ where $0\le x_0\le x_1\le \dots \le x_{N-1}\le 1$.

(Note that if $x_i\in[\frac{i}{N},\frac{i+1}{N}]$ then the Riemann sum is $\sum_{i=0}^{N-1} f(x_i)\left(\frac{i}{N}-\frac{(i-1)}{N}\right)=\frac{1}{N}\sum_{i=0}^{N-1} f(x_i)$).

Motivating Question: How to choose $\{x_i\}_{i=0}^{N-1}$ s.t. this is the best approximation for as many f as possible.

We want that $\left|\frac{1}{N}\sum_{i=0}^{N-1} f(x_i)-\int_0^1 f\right|\xrightarrow{N\to\infty}0$. By RI, this is true.

We want to study the rate of convergence. We get that $\left|\sum_{i=0}^{N-1} f(x_i)-N\int_0^1 f(x)\right|=o(N)$.

Denote: $D(\{x_i\}_{i=0}^{N-1},f):=\left|\sum_{i=0}^{N-1} f(x_i)-N\int_0^1 f(x)dx\right|$. If $\mathcal{F}$ is a collection of functions, $D(\{x_i\}_{i=0}^{N-1},\mathcal{F})=\sup_{f\in\mathcal{F}} D(\{x_i\}_{i=0}^{N-1},f)$, and $D(N,\mathcal{F})=\inf_{\{x_i\}_{i=0}^{N-1}} D(\{x_i\}_{i=0}^{N-1},\mathcal{F})$.

Q1 (naive): Is there a "universal sampling method", i.e. ~~a choice of points $\{x_i\}_{i=0}^{N-1}$ depending on N, s.t.~~ $\forall\varepsilon>0\ \exists N_0\ \forall N\ge N_0\ \exists\{x_i\}_{i=0}^{N-1}$ st $\forall f$ R.I. $D(\{x_i\}_{i=0}^{N-1},f)<\varepsilon N$?

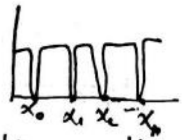
i.e., if we set $\mathcal{F}=\{f \text{ R.I.}\}$, does $D(N,\mathcal{F})=o(N)$?

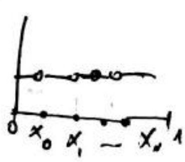
The answer is clearly NO, because if for any $\{x_i\}_{i=0}^{N-1}$ we take any f s.t. $D(\{x_i\},f)=\delta>0$, then $D(\{x_i\},\frac{N}{\delta}f)=N$, which is large.

Q2 (naive) $\forall \varepsilon>0\ \exists N_0\ \forall N\geq N_0\ \exists\{x_i\}$ s.t. $\forall f$ R.I. $D(\{x_i\},f) < \varepsilon N\|f\|_\infty$?

i.e. $\mathcal{F}=\left\{\begin{matrix} f \text{ is R.I.} \\ \|f\|_\infty=1\end{matrix}\right\}$, $D(N,\mathcal{F})=o(N)$?

This is still false: Given $\{x_i\}_{i=0}^{N-1}$ define $f(x)=\begin{cases}0 & x=x_i\\ 1 & \text{otherwise}\end{cases}$, and then $D(\{x_i\},f)=N$. This example can obviously be made continuous, so this is still false for $\mathcal{F}=\left\{\begin{matrix} f\in C[0,1] \\ \|f\|_\infty=1\end{matrix}\right\}$.





Thus we need to take smaller $\mathcal{F}$ to make the question sensible.

In the literature, two choices of $\mathcal{F}$ has been ~~studied~~ extensively studied:

① $\mathcal{F}=R_1=\{1_{[a,b]},\ 0\leq a\leq b\leq 1\}$

② Smooth Functions: $C^k([0,1])$. There is a Sobolev Norm: $\|f\|_{k,p}=\max(\|f\|_p,\|f'\|_p,\dots,\|f^{(k)}\|_p)$.

In particular, $\|f\|_{1,\infty}=\max(\|f\|_\infty,\|f'\|_\infty)$.

Sensible Question: $\mathcal{F}=\{f\in C^1([0,1]),\ \|f\|_{1,\infty}=1\}$ (now we can't get the example from before) and try to prove $D(N,\mathcal{F})=h(N)$ for some explicit h that satisfies $h(N)=o(N)$, i.e. find $h(N)$ s.t. $\forall N\ \exists\{x_i\}_{i=0}^{N-1}$ s.t. $\forall f\in C^1([0,1])$ $D(\{x_i\},f)\leq h(N)\|f\|_{1,\infty}$

A similar question for dim=2: $\mathcal{U}^d=[0,1]^d$, for $f:\mathcal{U}^d\to\mathbb{R}$, $\{x_i\}_{i=0}^{N-1}\subset\mathcal{U}^d$, define $D(\{x_i\}_{i=0}^{N-1},f)=\left|\sum f(x_i)-N\int_{\mathcal{U}^d}f\right|$. Define $R_2=\left\{\begin{matrix}\text{indicators of axis parallel}\\ \text{rectangles in } \mathcal{U}^2\end{matrix}\right\}=$

$=\{1_{[a_1,b_1]\times[a_2,b_2]}\}$.

Q: Find bounds on $D(N,R_2)$.

Easy proposition: $D(N,R_1)=O(1)$.

However, Thm: $\exists c_1,c_2>0$ s.t. $c_1\log N\leq D(N,R_2)\leq c_2\log N$.

Proof of prop.: For each N define $x_i=\frac{i}{N}$, $i=0,\dots,N-1$. Given $a<b$, ~~there are i,j s.t. $x_i<a<x_{i+1}$~~ take $i=\lfloor aN\rfloor$, $j=\lfloor bN\rfloor$. Then $\frac{j-i-2}{N}\leq b-a=\int_0^1 1_{[a,b]}\leq\frac{j-i+2}{N}$ and $\#\{i\mid x_i\in[a,b]\}\in(j-i-2,j-i+2)$, so we got Q.E.D.

History of the thm: Van der Corput - "Just distribution" question →

Van-Aardene Ehrenfest Thm (1945): $D(N,R_2)$ is not bounded below

(1949) $D(N,R_2)\geq\frac{c\log\log N}{\log\log\log N}$

Roth (1954) $D(N,R_2)\geq c\sqrt{\log N}$

Schmidt (1972) $D(N,R_2)\geq c\log N$,

The upper bound was known in the 1920's, with an explicit construction.

General Questions and Results of this type:

Given d and a collection $\mathcal{S}$ of subsets of $\mathcal{U}^d$, bound $D(N,\mathcal{F})$ where $\mathcal{F} = \{$indicators on sets from $\mathcal{S}\}$.

(one can also replace $\mathcal{U}^d$ with other subsets of $\mathbb{R}^d$).

Notation: R_d = axis parallel boxes in $\mathcal{U}^d$.

Thm: $c_1(\log N)^{\frac{d-1}{2}} \le D(N,R_d) \le C_d(\log N)^{d-1}$.

Closing the gap in this problem has been called "the great open problem in discrepancy".

- $\mathcal{S}_1^{(d)} = \{\mathcal{U}^{(d)}\cap B(x,r) \mid r>0,\ x\in\mathbb{R}^d\}$. **Thm:** $\forall d\ \forall\varepsilon\ \exists c_1$ $D(N,\mathcal{S}_1^{(d)}) \ge c_1 N^{\frac12-\frac1{2d}-\varepsilon}$

and $\exists c_2$ s.t. $D(N,\mathcal{S}_1^{(d)}) \le c_2 N^{\frac12-\frac1{2d}}\sqrt{\log N}$

- $\mathcal{S}_2 = \{\mathcal{U}^2\cap\{x: f^*(x)\ge c\} \mid f^*:\mathbb{R}^2\to\mathbb{R}$ is linear$\}$

Thm: $\exists c_1,c_2$ $c_1 N^{\frac14} \le D(N,\mathcal{S}_2) \le c_2 N^{\frac14}$

- Discrepancy of Rotated cubes: $\mathcal{S}_3 = \{\mathcal{U}^2\cap g(\mathcal{U}^2) \mid g$ is a similarity, i.e. $g=\lambda O(x)+y$, $\lambda>0$, $y\in\mathbb{R}^2$, $O\in O(2)\}$

λ is called the dilation of g.

~~Thm:~~

$\mathcal{S}_3(R) = \{\mathcal{U}^2\cap g(\mathcal{U}^2) : g$ is a similarity map with dilation in $[R,2R]\}$.

Thm: $\exists c$ s.t. If $R\in[\frac{1}{\sqrt N},\frac12)$ then $D(\mathcal{S}_3(R),N) \ge c N^{\frac14}R$.

Cor: $D(N,\mathcal{S}_3)\ge c N^{\frac14}$. (take $R=\frac12$).

We'll also study VC dimensions, to get probabilistic constructions for upper bounds.

What we discussed before could be called "static discrepancy". In "dynamic discrepancy" one fixes a sequence of samples $\{x_i\}_{i=0}^{\infty}$, and $\mathcal{F}$, and wants to bound $D(\{x_i\}_{i=0}^{N},\mathcal{F})$.

Example: Based on the observation that equispread points satisfy $D(\{x_i\}_{i=0}^{N},\mathcal{F}) = O(1)$, we want to construct an infinite sequence which is "often" equidistributed.

e.g. $(\frac12, 1, \frac14, \frac34, \frac18, \frac38, \frac58, \frac78, \dots)$ works for $N=2^n$, but for $N=\frac34\cdot 2^n$ it can be seen (exercise) that this is bad: $D(\{x_i\}_{i=0}^{N-1}, \mathbb{1}_{[0,\frac12]}) \ge \frac16 N \ne o(N)$.

Quantitative Ergodic Theorems

A p.p.s. (probability preserving system) is $(X,\mathcal{B},\mu,T)$ where X is a set, $\mathcal{B}\subseteq 2^X$ a σ-algebra, $\mu:\mathcal{B}\to[0,1]$ a probability measure, and T is a preserving measure transformation: (1) $T^{-1}(\mathcal{B})\subseteq\mathcal{B}$

(2) $A\in\mathcal{B}\Rightarrow \mu(T^{-1}(A)) = \mu(A)$

T is said to be <u>ergodic</u> if for any A satisfying $A = T^{-1}(A)$, $\mu(A) \in \{0,1\}$. (equivalently, whenever $\mu(A \Delta T^{-1}(A)) = 0 \Rightarrow \mu(A) \in \{0,1\}$).

<u>Motivation for ergodicity</u>: If $\mu(A) \in (0,1)$, $A = T^{-1}(A)$, one can form a new p.p.s by considering the restriction of T to A and $X \setminus A$.

<u>Birkhoff Ergodic Thm</u>: If $(X, \mathcal{B}, \mu, T)$ is an ergodic p.p.s. and $f \in L^1(X, \mu)$, then for μ-a.e. $x \in X$, (*) $\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) \xrightarrow[N\to\infty]{} \int_X f \, d\mu$.

$$(*) \Leftrightarrow \left| \sum_{n=0}^{N} f(T^n x) \xrightarrow[N\to\infty]{} N\int_X f \, d\mu \right| = o(N)$$

<u>Example</u> (irrational rotations) $X = S^1 \simeq \mathbb{R}/\mathbb{Z} \simeq \mathbb{T}^1$. We have the Borel σ-algebra $\mathcal{B}$
$$e^{2\pi i x} \mapsto x \bmod \mathbb{Z}$$
and the lebesgue measure μ. Take $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and define $T(x) = x + \alpha \pmod{\mathbb{Z}}$.

<u>Fact</u>: T is ergodic. ~~Thus~~ What does the Birkhoff theorem tells us in this example?

Take $f = \mathbb{1}_{[a,b)}$. $\int_X f \, d\mu = b-a$. Then for a.e. x, (*) holds. Let's take $x=0$:
$0 \le a < b < 1$)

$$\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) = \frac{1}{N}\#\{n \in \{0,\dots,N-1\} \mid T^n x \in [a,b)\} = \frac{1}{N}\#\{n \in \{0,\dots,N-1\} \mid n\alpha \bmod \mathbb{Z} \in [a,b)\}$$

<u>Fact</u>: In this case, one can use <u>all</u> x for any R.I. function (and not just almost all x). This follows from a stronger property called <u>unique ergodicity</u>.

Thus ($n\alpha \bmod \mathbb{Z}$) give low discrepancy for R.I. functions. An <u>effective ergodic theorem</u> is a theorem of the following form: Let $\mathcal{F}$ be a collection of functions on X, than there is an explicit rate function $E(n)$ and an explicit set of points $X_0 \subset X$ (T-invariant and full measure) s.t. for all $x \in X_0$ and $f \in \mathcal{F}$, $\left|\sum_{n=0}^{N-1} f(T^n x) - N\int_X f \, d\mu\right| \le E(N)$.

<u>Thm (Weyl, Behnke '22)</u>: If α is a quadratic irrational (e.g. $\sqrt{2}$ or $\varphi = \frac{1+\sqrt{5}}{2}$) and $\mathcal{F} = \left\{\begin{matrix}\text{indicators of} \\ \text{intervals}\end{matrix}\right\}$ then for all $f \in \mathcal{F}$, $x \in S$, $\left|\sum_{n=0}^{N-1} f(T^n x) - N\int_X f \, d\mu\right| \le C(\alpha) \log N$.

(This also works for badly approximable α). This is optimal for dynamic discrepancy: For any infinite sequence $\{x_i\} \subset U$, for $\mathcal{F} = \left\{\begin{matrix}\text{indicators of} \\ \text{intervals}\end{matrix}\right\}$, ~~there are inf~~ along a sequence $N_k \nearrow \infty$, $D(\{x_i\}_{i=0}^{N_k - 1}, \mathcal{F}) \ge c \log N_k$.

There is a reduction from dynamical discrepancy to statical, and this result is deduced from Schmidt's theorem ($D(N, R_2) \gtrsim \log N$).

An example we will discuss (time providing) is horocycle flow on compact quotients $SL_2(\mathbb{R})/\Gamma$.

#2
7.11.16

<u>Notation (Reminder)</u>: $\mathcal{U}^d=[0,1]^d$, $\mathcal{R}_d=\left\{\begin{matrix}\text{axis-parallel boxes}\\ \text{in } \mathcal{U}^d\end{matrix}\right\}=\left\{[a_1,b_1)\times\dots\times[a_d,b_d) : \begin{matrix}0\le a_i<b_i\le 1\\ i=1,\dots,d\end{matrix}\right\}$.

If $\{x_n\}_{n=0}^{N-1}$ is a sequence of points in $\mathcal{U}^d$, $f:\mathcal{U}^d\to\mathbb{R}$, we have $D((x_n)_{n=0}^{N-1};f)=\sum_{n=0}^{N-1}f(x_n)-N\int_{\mathcal{U}^d}f(x)\,dx$. If $\mathcal{F}$ is a collection of functions, then $D((x_n)_{n=0}^{N-1};\mathcal{F})=\sup_{f\in\mathcal{F}}|D((x_n)_{n=0}^{N-1};f)|$. If $\mathcal{S}$ is a collection of subsets of $\mathcal{U}^d$, $D((x_n)_{n=0}^{N-1};\mathcal{S})=D((x_n)_{n=0}^{N-1};\{1_A | A\in\mathcal{S}\})$. Classically one studies $D((x_n)_{n=0}^{N-1};\mathcal{R}_d)=\sup_{B \text{ axis parallel box}}|\#\{i<N | x_i\in B\}-N\,\mathrm{vol}(B)|$.

<u>"Static Discrepancy" Problem</u>: For each fixed N, choose $(x_i)_{i=0}^{N-1}$ s.t. $D((x_n);\mathcal{R}_d)$ is small.

<u>"Dynamical Discrepancy"</u>: We look for a fixed infinite sequence $(x_n)_{n=0}^{\infty}$ in $\mathcal{U}^d$ s.t. for all N, $\Delta_N((x_n);\mathcal{R}_d)=D((x_n)_{n=0}^{N-1};\mathcal{R}_d)$ is small.

We will see that studying static discrepancy for axis parallel boxes in $\mathbb{R}^d$ is equivalent to studying dynamical discrepancy for axis-parallel boxes in $\mathbb{R}^{d-1}$.

<u>Proposition</u>: $g:\mathbb{N}\to\mathbb{R}_+$ is non-decreasing, $d\ge 2$. Then TFAE (the following are equivalent):

(1) $\forall N\ \exists(x_n)_{n=0}^{N-1}$ in $\mathbb{R}^d$ s.t. $D((x_n)_{n=0}^{N-1};\mathcal{R}_d)=O(g(N))$

(2) $\exists(y_n)_{n=0}^{\infty}$ in $\mathbb{R}^{d-1}$ s.t. $\forall N$ $\Delta_N((y_n);\mathcal{R}_{d-1})=O(g(N))$.

<u>Proof</u>: Let $\mathcal{R}_d^*=\left\{\begin{matrix}\text{axis parallel boxes in } \mathcal{U}^d\\ \text{with a vertex at the origin}\end{matrix}\right\}=\left\{[0,b_1)\times\dots\times[0,b_d) \,\middle|\, \begin{matrix}0\le b_i<1\\ i=1,\dots,d\end{matrix}\right\}$.

<u>Lemma</u>: For each d, there is c_d s.t. for any $(x_i)_{i=0}^{N-1}$,
$$D((x_i)_{i=0}^{N-1};\mathcal{R}_d^*)\le D((x_i)_{i=0}^{N-1};\mathcal{R}_d)\le c_d\,D((x_i)_{i=0}^{N-1};\mathcal{R}_d^*)$$

<u>Proof</u>: In $d=1$, an element of $\mathcal{R}_d$ is an interval $[a,b)$ and of $\mathcal{R}_d^*$ is an interval $[0,b)$. Note that $[a,b)=[0,b)\setminus[0,a)$. So $\#\{i<N | x_i\in[a,b)\}=\#\{i<N | x_i\in[0,b)\}-\#\{i<N | x_i\in[0,a)\}$, and $\mathrm{vol}([a,b))=\mathrm{vol}([0,b))-\mathrm{vol}([0,a))$. Thus $D((x_i)_{i=0}^{N-1};[a,b))=D((x_i)_{i=0}^{N-1};[0,b))-D((x_i)_{i=0}^{N-1};[0,a))$.

Taking absolute values and supremum we see that $D((x_i);\mathcal{R}_1)\le 2D((x_i);\mathcal{R}_1^*)$, so take $c_1=2$.

(Note that the other inequality is trivially true as $\mathcal{R}_d^*\subseteq\mathcal{R}_d$).

For $d=2$, we'll draw a picture:

Thus $\#\{i<N | x_i\in R\}=\{i<N | x_i\in A\}-\{i<N | x_i\in B\}-\{i<N | x_i\in C\}+\{i<N | x_i\in D\}$.
$\mathrm{Vol}(R)=\mathrm{vol}(A)+\mathrm{vol}(D)-\mathrm{vol}(B)-\mathrm{vol}(C)$.

We get that $c_2=4$. For general d - exercise. Q.E.D. ☒

Now return to the proof of the proposition. By the lemma, we can work with $\mathcal{R}_d^*$.

We'll prove (2)⇒(1):

Given $(y_n)_{n=0}^{\infty}$ with $D((y_n)_{n=0}^{N-1}; R_{d-1}) = O(g(N))$ for all N, and given N define $x_i = (\underset{\in \mathcal{U}^{d-1}}{y_i}, \underset{\in \mathcal{U}^1}{\tfrac{i}{N}}) \in \mathcal{U}^d$.

$\#\{i<N \mid x_i \in [0,b_1)\times\dots\times[0,b_d)\} = \#\{i<N \mid y_i \in [0,b_1)\times\dots\times[0,b_{d-1}), \frac{i}{N} < b_d\} =$

$= \#\{i < N b_d \mid y_i \in [0,b_1)\times\dots\times[0,b_{d-1})\} = \#\left\{i \,\middle|\, \begin{matrix} i < \lfloor b_d N \rfloor \\ y_i \in [0,b_1)\times\dots\times[0,b_{d-1}) \end{matrix}\right\} =$

$= \lfloor b_d N \rfloor \mathrm{Vol}([0,b_1)\times\dots\times[0,b_{d-1})) + O(g(\lfloor b_d N \rfloor)) =$

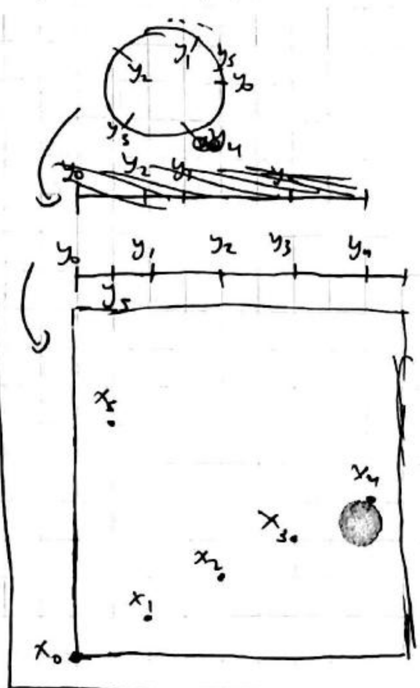
$\underset{n}{=} \left(b_d N \prod_{i=1}^{d-1} b_i + O(1)\right) + O(g(N)) =$

g is non-decreasing
$\lfloor b_d N \rfloor \le b_d N + 1$

$= N \prod_{i=1}^{d} b_i + O(g(N)) = N \mathrm{vol}([0,b_1)\times\dots\times[0,b_d)) + O(g(N))$.

$(1) \Rightarrow (2)$: Exercise.

Picture we take y_i generated by an irrational rotation



We will now study $\Delta_N((x_n); R_1)$. We want to get sequences with good discrepancy of the form $x_n = n\alpha \mod \mathbb{Z}$, $\alpha \in \mathbb{R}\setminus\mathbb{Q}$.

We will start with a qualitative fact.

Definition: A sequence $(x_n)_{n=0}^{\infty}$ in $\mathcal{U}^1$ is equidistributed if for any interval $I = [a,b] \subseteq \mathcal{U}^1$,

$\frac{1}{N}\#\{n<N \mid x_n \in I\} \xrightarrow{N\to\infty} \mathrm{Vol}(I) = b-a$.

Thm 1: $x_n = n\alpha \mod \mathbb{Z}$, $\alpha \in \mathbb{R}\setminus\mathbb{Q} \Rightarrow (x_n)$ is equidistributed.

We will use Thm 2 (Weyl Criterion, ~1910): (x_n) is equidistributed in $\mathcal{U}^1 \Leftrightarrow \forall h \in \mathbb{Z}, h \neq 0$, $\frac{1}{N}\sum_{n=0}^{N-1} e(hx_n) \xrightarrow[N\to\infty]{} 0$

(where $e(x) = e^{2\pi i x}$).

Thm 2 $\Rightarrow$ Thm 1: $x_n = n\alpha \pmod{\mathbb{Z}}$. Let $h \neq 0 \in \mathbb{Z}$. $\left|\frac{1}{N}\sum_{n=0}^{N-1} e(hx_n)\right| = \left|\frac{1}{N}\sum_{n=0}^{N-1} e(hn\alpha)\right| = \left|\frac{1}{N}\sum_{n=0}^{N-1} e(h\alpha)^n\right| =$

$= \left|\frac{1}{N} \frac{e(Nh\alpha)-1}{e(h\alpha)-1}\right| \le \frac{1}{N} \cdot \frac{2}{\underbrace{|e(h\alpha)-1|}_{\neq 0 \text{ since } \alpha \notin \mathbb{Q}}} \xrightarrow{N\to\infty} 0$

Proof of Thm 2: Let $\mathcal{F}$ denote all functions $f: \mathcal{U}^1 \to \mathbb{C}$ s.t. $\frac{1}{N}\sum_{n=0}^{N-1} f(x_n) \xrightarrow{N\to\infty} \int_0^1 f(x)\,dx$.

So $\mathcal{F}$ is a vector space over $\mathbb{C}$. And if $f_j \in \mathcal{F}$, $\|f_j - f\|_\infty \xrightarrow{j\to\infty} 0 \Rightarrow f \in \mathcal{F}$. (i.e. $\mathcal{F}$ is closed w.r.t. $\|\cdot\|_\infty$). (This is easy). If $f: \mathcal{U}^1 \to \mathbb{R}$ s.t. for every $\varepsilon > 0$, there are $f_1, f_2 \in \mathcal{F}$ s.t. $f_1(x) \le f(x) \le f_2(x)$, $\int f_2(x) - f_1(x) < \varepsilon$

then $f \in \mathcal{F}$. This is true because $\left|\int_0^1 f(x) - \int_0^1 f_i(x)\right| < \varepsilon$, $i = 1,2$, and $\limsup \frac{1}{N}\sum_{n=0}^{N-1} f(x_n) \le \limsup \frac{1}{N}\sum f_2(x_n) \to \int_0^1 f_2(x)$

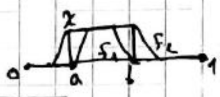
$\Rightarrow \limsup_{N\to\infty} \frac{1}{N}\sum f(x_n) \le \int_0^1 f(x)\,dx$, similarly for lim inf. We'll now prove the theorem.

$(\Rightarrow)$ Suppose $\chi_{[a,b)} \in \mathcal{F}$ for each $[a,b) \subseteq \mathcal{U}^1$, we want to show that $x \mapsto e(hx) \in \mathcal{F}$ for each $h \in \mathbb{Z}$.

(Note that for $h=0$ this works). Recall the definition of Riemann integrable real-valued functions:

f is R.I. is $\forall \varepsilon > 0\ \exists f_1, f_2$ step-functions s.t. $f_1(x) \le f(x) \le f_2(x)$ and $\int_0^1 f_2 - f_1 < \varepsilon$.

By the properties of $\mathcal{F}$, all R.I. functions belong to $\mathcal{F}$ thus $e(hx) = \cos(2\pi hx) + i\sin(2\pi hx) \in \mathcal{F}$.

(⟸): If $\mathcal{F}$ contains all the functions $x \mapsto e(hx)$, $h \in \mathbb{Z}$, then it contains all trigonometric ~~func~~ polynomials. The Stone-Weierstrass theorem says that trigonometric polynomials are dense in all continuous $\{f : \mathbb{T}^1 \to \mathbb{C}\}$ (w.r.t. $\|\cdot\|_\infty$). So all continuous functions belong to $\mathcal{F}$. Now, given $[a,b)$ in $\mathbb{T}^1$, and $\varepsilon > 0$, there are two continuous functions f_1, f_2 with $f_1 \le \chi_{[a,b)} \le f_2$ for all x, and $\int_0^1 (f_2 - f_1) < \varepsilon$. ( - this is an example) Q.E.D.

More common and general definition of equidistribution: Let X be a locally-compact separable metrizable space, $\mathcal{B}$ - the Borel σ-algebra, μ - a Borel probability measure, $(x_n)_{n=0}^{\infty}$ a sequence in X. We say that (x_n) is equidistributed w.r.t μ if for any continuous and compactly supported function $f : X \to \mathbb{R}$ (i.e. $\{x \mid f(x) \neq 0\}$ is compact, notation: $C_c(X)$), $\frac{1}{N}\sum_{n=0}^{N-1} f(x_n) \xrightarrow[N\to\infty]{} \int_X f\,d\mu$.

Our next goal is to prove effective bounds on discrepancy of $x_n = n\alpha$, $\alpha \in \mathbb{R}\setminus\mathbb{Q}$.

The following is known: For ~~a.e. α~~, If we set $x_n = n\alpha \bmod \mathbb{Z}$, then $\overbrace{D((x_n)_{n=0}^{\infty}; R_1)}^{\text{sometimes}} = O(\log N)$ (*).

We will prove (*) for α which are badly approximable. α is called badly approximable if $\exists c > 0\ \forall p \in \mathbb{Z}\ \forall q \in \mathbb{N}$, $|q\alpha - p| \ge \frac{c}{q}$. For example, any quadratic irrational is badly approximable.

~~We will prove~~ ~~Note that~~ $\cancel{|\alpha - \frac{p}{q}| \ge \frac{c}{q^2} \iff |q\alpha}$ Note that $\forall p,q\ |\alpha - \frac{p}{q}| \ge \frac{c}{q^2} \iff \forall p,q\ |q\alpha - p| \ge \frac{c}{q} \iff$

$\iff \forall q\ \langle q\alpha \rangle \ge \frac{c}{q}$ (where $\langle x \rangle = \operatorname{dist}(x, \mathbb{Z})$) $\iff \inf_{q \in \mathbb{N}} q\langle q\alpha\rangle > 0$. (*) is not true for a.e. α.

We will prove the Erdős-Turan inequality, which gives discrepancy bounds for all sequences (not just $x_n = n\alpha$) and gives $O((\log N)^2)$. Then we will use Denjoy-Koksma ineq. + Ostrowski expansion to prove (*) for badly approximable α. ~~We will not prove (*) for a.e. α.~~

Remark: For a.e. α, $\forall \varepsilon$ $D((n\alpha)_{n=0}^{N-1}; R_1) = O(\log N (\log\log N)^{1+\varepsilon})$.

Erdős-Turan Inequality

~~Suppose $x_n = n\alpha \bmod \mathbb{Z}$. Then~~ For any $(x_n)_{n=0}^{\infty}$ $\forall N, K \in \mathbb{N}$, $D((x_n)_{n=0}^{N-1}; R_1) = O\left(\frac{N}{K} + \sum_{h=1}^{K} \frac{1}{h}\left|\sum_{n=0}^{N-1} e(hx_n)\right|\right)$.

Proof that if α is B.A. then $x_n = n\alpha \bmod \mathbb{Z}$ satisfies $D(x_n; R_1) = O((\log N)^2)$:

We'll use Erdős-Turan with $K = N$. We need to show that $\sum_{h=1}^{N} \frac{1}{h}\left|\sum_{n=0}^{N-1} e(hx_n)\right| = O((\log N)^2)$.

As before, $\left|\sum_{n=0}^{N-1} e(hx_n)\right| = \left|\frac{e(Nh\alpha) - 1}{e(h\alpha) - 1}\right| \le \frac{2}{|e(h\alpha) - 1|}$. Note that $|e^{2\pi i h\alpha} - 1|$ is bounded above and below by $c_1\langle h\alpha\rangle$ and $c_2\langle h\alpha\rangle$ where $0 < c_1 < c_2$, so $\frac{2}{|e(h\alpha) - 1|} = O\left(\frac{1}{\langle h\alpha\rangle}\right)$.

So we need to show $\sum_{h=1}^{N} \frac{1}{h\langle h\alpha\rangle} = O((\log N)^2)$. First, let's bound $\sum_{h=1}^{j} \frac{1}{\langle h\alpha\rangle}$.

Claim: $\sum_{j=1}^{h} \frac{1}{\langle j\alpha\rangle} = O(h\log h)$.

~~Proof: Within the sequence $j\alpha \bmod 1$, $j=1,\dots,h$, Let j_0 be s.t. $\langle j_0\alpha\rangle$ is the smallest~~

~~Now, $\langle j_0\alpha\rangle \geq \frac{c}{j_0}$ since α is B.A. If $1\leq p<q\leq h$,~~

Proof: Choose j_0 s.t. $\langle j_0\alpha\rangle$ is the smallest. Now $\langle j_0\alpha\rangle \geq \frac{c}{j_0}$ since α is B.A..

If $1\leq p<q\leq h$, $|\langle q\alpha\rangle - \langle p\alpha\rangle| = \langle (q-p)\alpha\rangle \geq \frac{c}{j_0}$. So among the $\langle j\alpha\rangle$, there's only one in each interval $[0,\frac{c}{j_0}]$ $[\frac{c}{j_0}, \frac{2c}{j_0}]$, ... Then

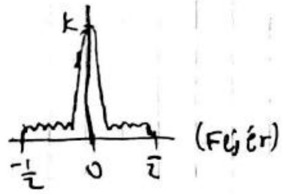
$$\sum_{j=1}^{h}\frac{1}{\langle j\alpha\rangle} \leq \frac{1}{\langle j_0\alpha\rangle} + \sum_{\substack{1\leq j\leq h\\ j\neq j_0}}\frac{1}{\langle j\alpha\rangle} \leq \frac{1}{\frac{c}{j_0}} + \sum_{j=1}^{h}\frac{1}{\frac{jc}{j_0}} \underset{(j_0\leq h)}{\leq} \frac{1}{c}\Big[j_0 + \sum_{j=1}^{h} h\cdot\frac{1}{j}\Big] \leq \frac{1}{c}\Big[h + \sum_{j=1}^{h} h\cdot \frac{1}{j}\Big] = O(h\log h).$$ ▨

$$\text{Now, } \sum_{h=1}^{N}\frac{1}{h\langle h\alpha\rangle} = \sum_{h=1}^{N}\frac{1}{h}\Big(\sum_{j=1}^{h}\frac{1}{\langle j\alpha\rangle} - \sum_{j=1}^{h-1}\frac{1}{\langle j\alpha\rangle}\Big) \underset{\text{Abel Summation}}{=} \frac{1}{N+1}\sum_{j=1}^{N}\frac{1}{\langle j\alpha\rangle} + \sum_{h=1}^{N}\frac{1}{h(h+1)}\sum_{j=1}^{h}\frac{1}{\langle j\alpha\rangle} =$$

$$= O\Big(\frac{1}{N}N\log N + \sum_{h=1}^{N}\frac{1}{h}\log h\Big) = O(\log N + (\log N)^2) = O((\log N)^2).\quad \text{Q.E.D.}$$

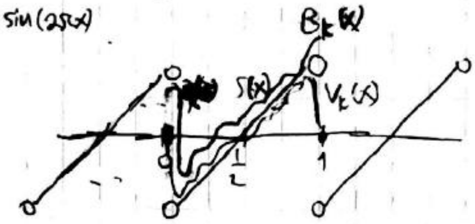
Proof of the Erdős-Turan Inequality: We want trigonometric polynomials of small degree bounding $\chi_{[a,b)}$ from above and below:

Fejér kernel: $F_k(x) = \sum_{|h|<k}\Big(1-\frac{|h|}{k}\Big)e(hx) = \frac{1}{k}\Big(\frac{\sin(\pi k x)}{\sin(\pi x)}\Big)^2$.



Vaaler's Function: $V_k(x) = \frac{1}{k+1}\sum_{h=1}^{k}\Big(\frac{h}{k+1}-\frac{1}{2}\Big)F_{k+1}\Big(x-\frac{h}{k+1}\Big) + \frac{1}{2\pi(k+1)}\sin(2\pi(k+1)x) - \frac{1}{2\pi}F_{k+1}(x)\sin(2\pi x)$.

"Sawtooth function": $s(x) = \begin{cases} x - \lfloor x\rfloor - \frac{1}{2} & x\notin\mathbb{Z}\\ 0 & x\in\mathbb{Z}\end{cases}$



Beurling function: $B_k(x) = V_k(x) + \frac{1}{2k+1}F_k(x)$

Lemma: If $0\leq x\leq \frac{1}{2}$ then $s(x)\leq V_k(x)\leq B_k(x)$. If $\frac{1}{2}\leq x\leq 1$ then $V_k(x)\leq s(x)\leq B_k(x)$. V_k, B_k are trig. polys. of degree k. If $T(x)$ is a trig. poly. of degree k s.t. $T(x)\geq s(x)$ for all x, then $\int_0^1 T(x)\,dx \geq \frac{1}{2(k+1)}$ with equality if and only if $T(x) = B_k(x)$.

If $I = [\alpha,\beta)$ then $\chi_I = \beta-\alpha + s(x-\beta) + s(\alpha - x)$. Define $S_k^+(x) = (\beta-\alpha) + B_k(x-\beta) + B_k(\alpha - x)$

$S_k^-(x) = (\beta-\alpha) - B_k(\beta - x) - B_k(x-\alpha)$.

Then $S_k^-\leq \chi_I \leq S_k^+$. $S^\pm$ are called the Selberg functions.

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Note that (1) $S_k^-\leq\chi_I\leq S_k^+$, (2) $\int_0^1 S_k^\pm(x) = b-a \pm \frac{1}{k+1}$.

We will prove the more precise:

$$(*)\quad \left| \#\{n<N \mid x_n \in [a,b]\} - N(b-a) \right| \le \frac{N}{k+1} + 2\sum_{h=1}^{k}\left(\frac{1}{k+1} + \min\left((b-a), \frac{1}{\pi h}\right)\right)\left|\sum_{n=0}^{N-1} e(hx_n)\right|.$$

<u>Reminder (Fourier Analysis)</u>: $\underset{[0,1)\to\mathbb{C}}{\overset{h\in\mathbb{Z}}{x\mapsto e(hx)}}$ are an orthogonal system for $\langle f,g\rangle = \int_0^1 f\bar{g}\,dx$. A function of the form $P(x) = \sum_{|h|\le k} a_h e(hx)$, where $a_k \ne 0$ or $a_{-k}\ne 0$ is called a trigonometric polynomial of degree k, and $P(x) = \sum_{|h|\le k} \hat{P}(h) e(hx)$ where $\hat{P}(h) = \langle P, e(hx)\rangle$. If μ is a measure, we say $\hat{\mu}(h) = \int e(-hx)\,d\mu$

<u>Proof of $(*)$</u>: $\#\{n<N \mid x_n\in[a,b]\} \le \sum_{n=0}^{N-1} S_k^+(x_n) = \sum_{n=0}^{N-1}\sum_{|h|\le k} \hat{S}_k^+(h) e(hx_n) = \sum_{|h|\le k} \hat{S}_k^+(h)\hat{\mathcal{U}}_N(-h) =$

where $\hat{\mathcal{U}}_N(h) = \sum_{n=0}^{N-1} e(-hx_n)$, which are the Fourier coefficients of $\mathcal{U}_N = \sum_{n=0}^{N-1}\delta_{x_n}$.

$$= \hat{S}_k(0)\hat{\mathcal{U}}_N(0) + \sum_{0<|h|\le k} \hat{S}_k(h)\hat{\mathcal{U}}_N(-h) \le N(b-a) + \frac{N}{k+1} + \sum_{0<|h|\le k} |\hat{S}_k(h)||\hat{\mathcal{U}}_N(-h)|$$

$$\hat{S}_k^+(0) = \int_0^1 S_k^{\pm} = b-a \pm \frac{1}{k+1}$$

Note that by rearranging we get something similar to $(*)$, we only need to bound $|\hat{S}_k^+(h)|$.

For any f and any h, $|\hat{f}(h)| \le \int_0^1 |f|\,dx = \|f\|_1$. We'll apply that to $f = S_k^+(x) - \chi_{[a,b]}(x)$.

As $f\ge 0$ we get $\|f\|_1 = (b-a) + \frac{1}{k+1} - (b-a) = \frac{1}{k+1}$. We need to understand $\hat{\chi}_{[a,b]}(h)$:

$$\hat{\chi}_{[a,b]}(h) = \int_a^b e(-hx)\,dx = \frac{e(-hb) - e(-ha)}{-2\pi h i} \Rightarrow |\hat{\chi}_{[a,b]}(h)| = \left|\frac{\sin \pi h(b-a)}{\pi h}\right| \le \min\left(b-a, \frac{1}{\pi|h|}\right).$$

So $|\hat{S}_k^+(h)| \le \|f\|_1 + |\hat{\chi}_{[a,b]}(h)| \le \frac{1}{k+1} + \min\left(b-a, \frac{1}{\pi|h|}\right)$. Combining all of that, we get QED.

<u>Remarks</u>: 1. In the proof, we could take a sequence of measures ν_N in place of $\mathcal{U}_N = \sum_{n=0}^{N-1}\delta_{x_n}$. LHS would be $\nu_N([a,b])$, and in the RHS we would use $\hat{\nu}_N(h)$ instead of $\hat{\mathcal{U}}_N(h)$. This gives a more general inequality with $|\hat{\mathcal{U}}_N(h)|$ replacing $\left|\sum_{n\ge 0}^{N-1} e(hx_n)\right|$!

2. Generalization to $\mathcal{U}^d$: We can think of $\mathcal{U}^d$ as $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Using $\underset{\underline{x}\in\mathcal{U}^d}{\underline{h}\in\mathbb{Z}^d}$, define $e(\underline{h}\,\underline{x}) = e^{2\pi i\langle \underline{x},\underline{h}\rangle}$.

$\{x\mapsto e(\underline{h}\,\underline{x})\}$ are an orthonormal family and a basis for L^2. We get that for $(\underline{x}_n)_{n=0}^{N-1}\subset\mathcal{U}^d$,

$$D\left((\underline{x}_n)_{n=0}^{N-1}; R_d\right) \le O_d\left(\frac{N}{k} + \sum_{0<\|\underline{h}\|_\infty\le k} \frac{1}{r(\underline{h})}\left|\sum_{n=0}^{N-1} e(\underline{h}\,\underline{x}_n)\right|\right)$$ where $\|h\|_\infty$ is the sup-norm and ~~[illegible]~~

(constants depend on d)

$r(\underline{h}) = \prod_{i=1}^{d}\max(1,|h_i|)$. The idea is that if $S_k^{1,+}, S_k^{2,+}$ are good approximations from above, for $\chi_{[a_1,b_1)}, \chi_{[a_2,b_2)}$ then $S_k^{1,+}\otimes S_k^{2,+}$ is a good approximation from above to $\chi_{[a_1,b_1)\times[a_2,b_2)}$. $((f\otimes g)(x,y) = f(x)g(y))$.

~~In the proof we used~~ $\cancel{(1)\ S_k^- \le \chi_{[a,b)} \le S_k^+ \quad (2) \int_0^1 S_k^{\pm} = b-a \pm \frac{1}{k+1}}$.

Last time we saw that using E-T we can show $D\left((n\alpha)_{n=0}^{N-1}; R_1\right) = O((\log N)^2)$ where $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ is B.A. (i.e. $|\alpha - \frac{p}{q}| \ge \frac{c}{q^2}$).

Thm 1: If α is badly approximable then $D((n\alpha)_{n=0}^{N-1}; R_1) = O(\log N)$ (the implicit constant in the O-notation depends on α, not on N).

Classical results in Diophantine approximations (Liouville) Every quadratic irrational is B.A. ~~Every~~

(Roth) Every algebraic irrational satisfies $\forall \varepsilon>0\ \exists c>0\ \forall p,q\ |x-\frac{p}{q}| \geq \frac{c}{q^{2+\varepsilon}}$

(Khinchin) If $\psi(q) \searrow 0$ "approximation fn", and $\sum q\psi(q)=\infty$ then for a.e. x $\exists$ inf. many p,q $|x-\frac{p}{q}|<\psi(q)$.
(this is sharp).

We will prove Thm 1.

Def: Let $f:[a,b)\to\mathbb{R}$. The variation of f is $Var(f) = \sup_{a=x_0<x_1<\dots<x_N<b} \sum_{n=0}^{N-1} |f(x_{n+1})-f(x_n)|$. If $Var(f)<\infty$ we say that f has bounded variation (BV). Sometime we will write $V_{ar}^{[a,b)}(f)$.

Example

(e.g. $\sin\frac{1}{x}$)

BV not BV

Note that if $a<b'<b$ and $f:[a,b]\to\mathbb{R}$ then $Var^{[a,b]}(f) = Var^{[a,b']}(f) + Var^{[b',b]}(f)$. (Ex)

If f is non-decreasing ~~If nondecreasing~~, $Var(f) = f(b)-f(a)$. If $f=f_1-f_2$, f_i non-decreasing, then $Var(f) = Var(f_1)+Var(f_2)$.

f is B.V. $\iff$ $\exists f_1,f_2$ non-decreasing with $f=f_1-f_2$.

Lemma Suppose f is B.V. on $[0,1]$, $(x_n)_{n=0}^{N-1} \subset [0,1]$, such that for each $0\le i<N$ there is one element of $(x_n)_{n=0}^{N-1}$ in $[\frac{i}{N}, \frac{i+1}{N})$ then $\left|\sum_{n=0}^{N-1} f(x_n) - N\int_0^1 f(x)\,dx\right| \le Var(f)$.

Proof: $\left|\sum_{n=0}^{N-1} f(x_n) - N\int_0^1 f(x)dx\right| = \left|\sum_{n=0}^{N-1} N\int_{\frac{n}{N}}^{\frac{n+1}{N}} [f(x_n)-f(x)]\,dx\right| \le \sum_{n=0}^{N-1} N\int_{\frac{n}{N}}^{\frac{n+1}{N}} |f(x_n)-f(x)|\,dx \overset{\text{since } x_n\in[\frac{n}{N},\frac{n+1}{N})}{\le} \sum_{n=0}^{N-1} \sup_{\frac{n}{N}\le x,y\le\frac{n+1}{N}} |f(x)-f(y)| \le$

$\le \sum_{n=0}^{N-1} Var^{[\frac{n}{N},\frac{n+1}{N}]}(f) \le Var(f)$. ∎

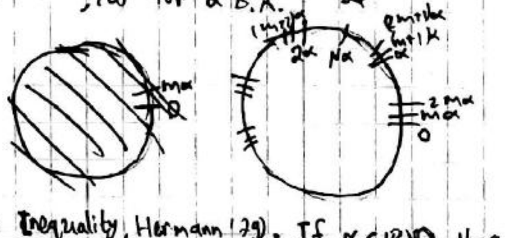
Steps in the Proof of Thm 1: Given α, we will define $q_n\nearrow\infty$ (convergents of α), satisfying

(3) for each n, if $N=q_n$ then $x_i=i\alpha$, $i=0,\dots,N-1$, satisfies the condition of the lemma, i.e. $[\frac{i}{N},\frac{i+1}{N})$ contains one of the x_i.

(4) any $N\in\mathbb{N}$ can be written as $N=\sum_{n=0}^{\infty} b_n q_n$, $b_n\ge 0$, $\sum_{n=0}^{\infty} b_n = O(\log N)$. (const depends on α, not on N).

Remark: (3) holds for all α, (4) for α B.A.

Picture for (3)



If $N\alpha$ is really close to 0, we have a problem as $(N+1)\alpha$ and so on are very close, so (3) might fail. Thus we want N to be a convergent, and those are the only obstructions.

Thm: (Denjoy-Koksma Inequality, Hermann '79). If $\alpha\in\mathbb{R}\setminus\mathbb{Q}$, $N=q_n$ is a convergent of α, $f\in BV$, then
$\left|\sum_{n=0}^{N-1} f(n\alpha) - N\int_0^1 f(x)dx\right| \le Var(f)$.

Proof: Lemma + (3). ∎

Cor: If $x_n = x_0+n\alpha$, for some x_0 and $N=q_n$ a convergent of α, then $|D((x_n)_{n=0}^{N-1}; R_1)| \le 2$.

Proof: $D((x_n)_{n=0}^{N-1}; R_1) = \sup_{0\le a<b\le 1} \left|\sum_{n=0}^{N-1} \chi_{[a,b]}(x_0+n\alpha) - N\int_0^1 \chi_{[a,b]}(x)\,dx\right| = \sup_{0\le a<b\le 1} \left|\sum_{n=0}^{N-1} \chi_{[a-x_0,b-x_0]}(n\alpha) - N\int_0^1 \chi_{[a-x_0,b-x_0]}(x)dx\right|$

$\leq Var(\chi_{[a-x_0, b-x_0]}) \leq 2$

$\chi_{[a-x_0, b-x_0]}$ is B.V., for piecewise const. functions f, $Var(f) = \sum_{\substack{f \text{ not} \\ \text{continuous at } x}} [|f(x+)-f(x)| + |f(x)-f(x-)|]$. ■

Proof of Thm 1 assuming (3),(4): Write $N = \underbrace{q_0 + \dots + q_0}_{b_0 \text{ times}} + \underbrace{q_1 + \dots + q_1}_{b_1 \text{ times}} + \dots$ where $\sum b_n = O(\log N)$ (using (4)). Denote $s_{ij} := b_0 q_0 + \dots + b_{i-1} q_{i-1} + j q_i$. For each $0 \leq a < b < 1$:

$$\left|\sum_{n=0}^{N-1} \chi_{[a,b)}(n\alpha) - N(b-a)\right| \leq \sum_{i=0}^{\infty} \sum_{j=0}^{b_i - 1} \left| \sum_{s_{ij} \leq n < s_{ij}+q_i} \chi_{[a,b)}(n\alpha) - q_i(b-a)\right| =$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{b_i-1} \left| \sum_{0 \leq m < q_i} \chi_{[a,b)}(\underbrace{s_{ij}\alpha}_{x_0} + m\alpha) - q_i(b-a)\right| \underset{\text{By the Cor.}}{\leq} \sum_{i=0}^{\infty} \sum_{j=0}^{b_i-1} 2 = 2\sum_{i=0}^{\infty} b_i = O(\log N). \text{ Q.E.D.}$$

Continued Fractions: Any α can be written as $\alpha = \lim_{n\to\infty} \left(a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}\right) = \lim_{n\to\infty} \frac{p_n}{q_n}$, $a_0 \in \mathbb{Z}, a_i \in \mathbb{N}$ $i \geq 1$.

q_n are called convergents. Notation: $\alpha = [a_0; a_1, a_2, \dots]$. a_i are called digits.

Thm: To each $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\exists (a_0, a_1, \dots)$ as above and p_n, q_n s.t.

ⓐ $\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$, $\gcd(p_n, q_n) = 1$, $\det\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1}\end{pmatrix} = (-1)^{n+1}$, $q_{n+1} = a_{n+1} q_n + q_{n-1}$, $p_{n+1} = a_{n+1} p_n + p_{n-1}$

$p_0 = a_0, q_0 = 1, p_{-1} = 1, q_{-1} = 0$

ⓑ For n odd, $\dots < \frac{p_{n-1}}{q_{n-1}} < \frac{p_{n+1}}{q_{n+1}} < \dots < \alpha < \dots < \frac{p_n}{q_n} < \frac{p_{n-2}}{q_{n-2}} < \dots$

ⓒ For each n, $\frac{1}{a_{n+1} q_n^2} > \left|\alpha - \frac{p_n}{q_n}\right| > \frac{1}{(a_{n+1}+2) q_n^2}$ (α is BA $\iff$ a_n are bounded)

ⓓ $\langle q_n \alpha \rangle < \min_{q < q_n} \langle q\alpha \rangle$, i.e. q_n's are the times when $q_n\alpha$ is closer to α then all predecessors. and if $\langle q'\alpha \rangle < \min_{q<q'} \langle q\alpha \rangle$ then $q' = q_n$ for some n.

ⓔ The map $cf: \left\{(a_0, a_1, \dots) \middle| \begin{array}{l} a_0 \in \mathbb{Z} \\ a_i \in \mathbb{N}\ i \geq 1\end{array}\right\} \to \mathbb{R}/\mathbb{Q}$ given by $cf((a_i)) = \lim_{n\to\infty} a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$ is well-defined and a bijection.

We will only need ⓐ,ⓑ,ⓒ, so we will prove only them.

Proof of (3) using ⓐ,ⓑ,ⓒ: By ⓒ, $\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}$. Assume $0 < \alpha - \frac{p_n}{q_n} < \frac{1}{q_n^2}$.

For $i = 0, \dots, q_n - 1$, we get that $0 < i\alpha - \frac{ip_n}{q_n} < \frac{i}{q_n^2} < \frac{1}{q_n}$. Write $ip_n = rq_n + l_i$, $l_i \in \{0, \dots, q_n - 1\}$.

Subtracting r: $\frac{l_i}{q_n} < i\alpha - \lfloor i\alpha \rfloor < \frac{l_i + 1}{q_n}$. As $\gcd(p_n, q_n) = 1$ (by ⓐ), $i \mapsto l_i$ is bijective. Thus each interval $[\frac{j}{N}, \frac{j+1}{N}]$ contains some $\{i\alpha\}$. ■

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Proof of (4) using ⓐ,ⓑ,ⓒ: Since α is B.A., $\sup_n a_n < \infty$. So $\exists c_1, c_2$ s.t. $0 < c_1 < \frac{q_{n-1}}{q_{n+1}} < c_2 < 1$.

Clearly $2q_{n-1} \leq q_{n+1}$ by ⓐ, and $q_{n+1} \leq (M+1) q_n \leq (M+1)^2 q_{n-1}$ where $M = \sup_n a_n$. So $\exists c_3, c_4$ s.t. $0 < c_3 < 1 - \frac{q_{n-1}}{q_{n+1}} < c_4 < 0$, and thus $\exists c_5 > 0$ s.t. $\log\left(1 - \frac{q_{n-1}}{q_{n+1}}\right) < -c_5$. Set $C = \frac{1}{c_5}$.

We'll prove that $\sum b_i \leq C \log(N) + 1$.

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~~By induction~~ By induction on N: It's clear for $N=1$. Now suppose it's true for all $N_0<N$.
Chose n s.t. $q_{n-1} < N \le q_{n}$. Let $N_0 = N - q_{n-1}$. Then $N_0 = \sum b_i q_i$ with $m = \sum b_i \le C\log N_0 + 1$ and N can be written with $m+1$ ~~by~~ instead of m. Then

$$m+1 \le C\log N_0 + 1 + 1 = 1 + C\log(N - q_{n-1}) + 1 = 1 + C\log\left(N\left(1-\frac{q_{n-1}}{N}\right)\right) + 1 = 1 + C\log N + C\log\left(1-\frac{q_{n-1}}{N}\right) + 1$$

$$< 1 + C\log N + C\log\left(1-\frac{q_{n-1}}{q_n}\right) + 1 \le 1 + C\log N - C\cdot c_5 + 1 \le 1 + C\log N. \quad \blacksquare$$

Remarks: ① Since $q_0=1$ any N can be written as $\sum b_i q_i$, $b_i\in\mathbb{N}$. There is a unique choice of the (b_i) which makes $\sum b_i$ smallest (and $\sum b_i q_i = N$) and this is called the Ostrowski Expansion of N. It is obtained by "greedy algorithm": given N, find n s.t. $q_{n-1} \le N < q_n$ and write an Ostrowski expansion for $N - q_{n-1}$, and add q_{n-1}.

② In some cases, one can get a more efficient representation of N using positive and negative coefficients, i.e. $N=\sum b_i q_i$ with $b_i\in\mathbb{Z}$, $\sum|b_i|$ small.

Proof of Thm on Continued fractions (a), (b), (c):

(a) $\frac{p_n}{q_n} = a_0 + \frac{1}{a_1 + \dots + \frac{1}{a_n}}$. Denote $\frac{\tilde p_{n-1}}{\tilde q_{n-1}} = a_1 + \frac{1}{a_2+\dots+\frac{1}{a_n}}$. Then $\frac{p_n}{q_n} = a_0 + \frac{1}{\frac{\tilde p_{n-1}}{\tilde q_{n-1}}} = a_0 + \frac{\tilde q_{n-1}}{\tilde p_{n-1}} =$

$$= \frac{a_0\tilde p_{n-1}+\tilde q_{n-1}}{\tilde p_{n-1}} \sim \begin{bmatrix}a_0&1\\1&0\end{bmatrix}\begin{bmatrix}\tilde p_{n-1}\\ \tilde q_{n-1}\end{bmatrix} \underset{\text{by induction}}{\sim} \begin{bmatrix}a_0&1\\1&0\end{bmatrix}\begin{bmatrix}a_1&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_{n-1}&1\\1&0\end{bmatrix}\begin{bmatrix}a_n\\1\end{bmatrix} =$$

$= \begin{bmatrix}a_0&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_n&1\\1&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$. We have proved that $\begin{bmatrix}p_n\\q_n\end{bmatrix} = \begin{bmatrix}a_0&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_n&1\\1&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$.

So $\begin{bmatrix}p_{n-1}\\q_{n-1}\end{bmatrix} = \begin{bmatrix}a_0&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_{n-1}&1\\1&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}a_0&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_{n-1}&1\\1&0\end{bmatrix}\begin{bmatrix}a_n&1\\1&0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$.

So $\begin{bmatrix}p_n&p_{n-1}\\q_n&q_{n-1}\end{bmatrix} = \begin{bmatrix}a_0&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_n&1\\1&0\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix} \Rightarrow \det\begin{pmatrix}p_n&p_{n-1}\\q_n&q_{n-1}\end{pmatrix} = (-1)^{(n+1)}$, and so p_n, q_n are coprime (and thus matrix multiplication gives the right p_n, q_n and not up to a multiple).

We get that $\begin{bmatrix}p_{n+1}&p_n\\q_{n+1}&q_n\end{bmatrix} = \begin{bmatrix}p_n&p_{n-1}\\q_n&q_{n-1}\end{bmatrix}\begin{bmatrix}a_{n+1}&1\\1&0\end{bmatrix}$, so we get $\blacksquare$.

(b) If n is even (resp. odd) then $b\mapsto a_0 + \frac{1}{a_1+\dots+\frac{1}{b}}$ is increasing (resp. decreasing).
Define a_n inductively by the following rule: For odd n, $\frac{p_{n-1}}{q_{n-1}} < \alpha < \frac{p_n}{q_n}$ so $\frac{p_{n+1}}{q_{n+1}} = a_0 + \frac{1}{a_1+\dots+\frac{1}{a_{n+1}}}$, choose a_{n+1} as large as possible such that $\frac{p_{n+1}}{q_{n+1}} < \alpha$. For n even, reverse the inequalities.

(c) $\left|\frac{p_{n+2}}{q_{n+2}} - \frac{p_n}{q_n}\right| \le \left|\alpha - \frac{p_n}{q_n}\right| \le \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right|$ (by (b)).

RHS: $\left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \frac{|p_{n+1}q_n - p_n q_{n+1}|}{q_n q_{n+1}} = \frac{1}{q_n q_{n+1}} = \frac{1}{q_n(a_{n+1}q_n+q_{n-1})} \le \frac{1}{a_{n+1}q_n^2}$

~~LHS:~~ $\cancel{\left|\frac{p_{n+2}}{q_{n+2}} - \frac{p_n}{q_n}\right| = \left|\frac{p_{n+2}q_n - p_n q_{n+2}}{q_n q_{n+2}}\right|}$

LHS $\left|\frac{p_{n+2}}{q_{n+2}} - \frac{p_n}{q_n}\right| = \left|\frac{p_{n+2}q_n - p_n q_{n+2}}{q_{n+2}q_n}\right| = \left|\frac{(a_{n+2}p_{n+1}+p_n)q_n - (a_{n+2}q_{n+1}+q_n)p_n}{q_n q_{n+2}}\right| =$

$$= \frac{a_{n+2}|p_{n+1}q_n - q_{n+1}p_n|}{|q_n q_{n+2}|} = \frac{a_{n+2}}{q_n(a_{n+2}q_{n+1}+q_n)} = \frac{1}{q_n\left(q_{n+1}+\frac{q_n}{a_{n+2}}\right)} > \frac{1}{q_n(q_{n+1}+q_n)} >$$

$$> \frac{1}{(a_{n+1}+2)q_n^2}. \ \blacksquare$$

Van der Corput Sequence

Recall! For static discrepancy, for fixed N, the sequence $0, \frac{1}{N}, \dots, \frac{N-1}{N}$ has $D((x_n)_{n=0}^{N-1}; \mathbb{R}_1) = O(1)$ and this is best possible. For dynamic discrepancy, one might want to choose $(x_n)_{n=0}^{\infty}$ s.t. for many N, $(x_n)_{n=0}^{N-1}$ is of the form above.

Example $0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \dots$ this doesn't work, but it can be fixed if we premute the blocks.

In base 10: $0.1, 0.2, \dots, 0.9, 0.01, 0.02, \dots, 0.09, 0.11, \dots, 0.99$

Remedy (VdC sequence in base 10)! $0, 0.1, \dots, 0.9, 0.01, 0.11, \dots, 0.91, 0.02, \dots, 0.92, \dots, 0.99$

$0, 0.1, \dots, 0.9,$
$0.01, 0.11, \dots, 0.91$
$0.02, 0.12, \dots, 0.92,$
$\dots\dots$ 0.99

In base 2: $0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \frac{1}{16}, \frac{9}{16}, \frac{5}{16}, \frac{13}{16}, \frac{3}{16}, \frac{11}{16}, \frac{7}{16}, \frac{15}{16}, \dots$

Formal Def! Fix a base $b \in \mathbb{N}, b \geq 2$. For $n = \sum_{i=0}^{\infty} a_i b^i$ (base b expansion, so $a_i \in \{0, \dots, b-1\}$), Let $x_n = \sum_{i=0}^{\infty} a_i b^{-(i+1)}$. (Also called "bit reversal sequence" for obvious reasons).

Example $b=2$, $n = 13 = 8+4+1$, $a_0 = 1, a_1 = 0, a_2 = 1, a_3 = 1$. So $x_{13} = \frac{1}{2} + \frac{1}{8} + \frac{1}{16} = \frac{11}{16}$.

Prop (Van der Corput, '40s): For fixed b, for any N, $D((x_n)_{n=0}^{N-1}; \mathbb{R}_1) = O(\log(N))$.

Remark! In the previous result, we want $\alpha \in BA$ with $\sup_n a_n$ as small as possible to get smallest ~~possible~~ discrepancy. Clearly M is the smallest exactly when $a_0 = a_1 = \dots = 1$. This gives $\alpha = \varphi = \frac{1+\sqrt{5}}{2} = [1; 1, 1, \dots]$.

Remark: The implicit constant in the prop. depends on b and not on N.

Pf of Prop: Recall that $D((x_n)_{n=0}^{N-1}; \mathbb{R}_1) = \sup_{I \subseteq \mathbb{U}^1} |\#\{n < N \mid x_n \in I\} - N\mathrm{Vol}(I)|$. In the proof we'll work with $b=2$ for simplicity.

Step 1: Let $I = [\frac{k}{2^q}, \frac{k+1}{2^q})$ for some $q \in \mathbb{N}$, $0 \leq k < 2^q$. All the x_n have a base 2 expansion. The ones which belong to I have the same expansion as k in the q most significant digit. These digits are the q least significant digits in the expansion of n.

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This means that there is some c s.t. $x_n \in I \iff n \equiv c \pmod{2^q}$. These n form an arithmetic progression of jumps 2^q. So there are $\lceil \frac{N}{2^q} \rceil$ or $\lfloor \frac{N}{2^q} \rfloor$ of them. ~~But Vol(I) = $\frac{1}{2^q}$, so~~

$\left|\lceil \frac{N}{2^q} \rceil - \frac{N}{2^q}\right|, \left|\lfloor \frac{N}{2^q} \rfloor - \frac{N}{2^q}\right| < 1$. But $Vol(I) = \frac{l}{2^q}$, so for such I, $D((x_n)_{n=0}^{N-1}; I) = O(1)$.

Step 2: Now suppose $I = [0, \frac{b}{2^m})$, $b \in 1, \dots, 2^m$, $2^m \le N < 2^{m+1}$. Using base 2-expansion of b, we can represent I as a finite disjoint union of intervals as in step 1, $I = I_1 \cup \dots \cup I_j$, $j \le \lceil \log_2 N \rceil$. So

$\left| \#\{n < N \mid x_n \in I\} - N Vol(I) \right| \le \sum_{k=1}^{j} \left| \#\{n < N \mid x_n \in I_k\} - N vol(I_k) \right| \le j\, O(1) = O(\log_2 N)$.

Step 3. For $I = [0, b)$, $b \in [0,1)$. Take $b' \in \{1, \dots, 2^m\}$ s.t. $\frac{b'}{2^m} < b < \frac{b'+1}{2^m}$, where $2^m \le N < 2^{m+1}$.

$\#\{n < N \mid x_n \in I\} - N Vol(I) \le \#\{n < N \mid x_n \in [0, \frac{b'+1}{2^m})\} - N\frac{b'}{2^m} \le N \frac{b'+1}{2^m} + O(\log N) - N\frac{b'}{2^m} =$

$= O(\log N) + \frac{N}{2^m} = O(\log N) + O(1) = O(\log N)$.

Similarly for $N Vol(I) - \#\{n < N \mid x_n \in I\}$.

Step 4 $D((x_n)_{n=0}^{N-1}; R_1) = O(D((x_n)_{n=0}^{N-1}; R_1^*))$ where $R_1^* = \{[0,b) \mid b \in [0,1)\}$. Q.E.D.

Let's formalize an idea which came up in the proof.

Prop: Let $\mathcal{S}$ be a collection of subsets of U^d, $\mathcal{S}' \subset \mathcal{S}$ finite. For any $A \in \mathcal{S}$, let

$\delta_{\mathcal{S}'}(A) = \min\left\{ Vol(A_2 \setminus A_1) \,\middle|\, \begin{matrix} A_1 \subset A \subset A_2 \\ A_1, A_2 \in \mathcal{S}' \end{matrix} \right\}$, $\delta_{\mathcal{S}'} = \sup_{A \in \mathcal{S}} \delta_{\mathcal{S}'}(A)$. Then for any $(x_n)_{n=0}^{N-1}$,

$\left| D((x_n)_{n=0}^{N-1}; \mathcal{S}) \right| \le \left| D((x_n)_{n=0}^{N-1}; \mathcal{S}') \right| + N\delta_{\mathcal{S}'}$.

Proof: Exercise. (similar to step 3).

Generalization to higher dimension.

Halton-Hammersley sequences ('60s)

Let $p_1, p_2, \dots, p_d$ be distinct primes (or integers satisfying $\gcd(p_i, p_j) = 1$, $i \ne j$).

For each p, define $r_p(n)$ as before using $b = p$ i.e. $r_p(a_0 + a_1 p + \dots) = \frac{a_0}{p} + \frac{a_1}{p^2} + \dots$.

The H-H sequence is $(r_{p_1}(n), \dots, r_{p_d}(n))_{n=0}^{\infty}$.

Prop For any $p_1, \dots, p_d$, and $d \ge 2$, any N, $D((x_n)_{n=0}^{N-1}; R_d) = O(\log(N)^d)$, where (x_n) is the H-H sequence.

Remark As before, implicit constant depends on $p_1, \dots, p_d$ (in particular on d).

Proof. (In the case $d=2, p_1=2, p_2=3$).

Step 1 Suppose $I = [\frac{k}{2^q}, \frac{k+1}{2^q}) \times [\frac{l}{3^r}, \frac{l+1}{3^r})$. Let's show $|D((x_n)_{n=0}^{N-1}; I)| \le 1$. As in the previous proof, there are c_1, c_2 s.t. $x_n \in I \iff \begin{matrix} n \equiv c_1 (2^q) \\ n \equiv c_2 (3^r) \end{matrix}$, so by the chinese remainder theorem, since $\gcd(2^q, 3^r) = 1$, all solutions n belong to an arithmetic progression of jumps $2^q 3^r$.

So $\left| \#\{n < N \mid x_n \in I\} - N Vol(I) \right| = \left| \left(\lfloor \frac{N}{2^q 3^r} \rfloor \text{ or } \lceil \frac{N}{2^q 3^r} \rceil \right) - \frac{N}{2^q 3^r} \right| < 1$.

Step 2 Suppose $I=[0,\frac{b_1}{2^{m_1}})\times[0,\frac{b_2}{3^{m_2}})$ where $2^{m_1}\le N<2^{m_1+1}$, $3^{m_2}\le N<3^{m_2+1}$.

$[0,\frac{b_1}{2^{m_1}})$ can be written as a disjoint union of intervals as in step 1, using base 2 expansion of b.
The number of intervals is at most m_1, since it is the sum of the digits in the expansion.
$[0,\frac{b_2}{3^{m_2}})$ can be written as a disjoint union of at most $2m_2$ intervals.
(e.g. $[0,\frac{5}{9})=[0,\frac{1}{3})\cup[\frac{1}{3},\frac{4}{9})\cup[\frac{4}{9},\frac{5}{9})$). Thus I is a union of at most $m_1\cdot 2m_2$ $O(\log N^2)$ boxes as discussed in step 1. (In general, for $p_1,\dots,p_d$, we get $\prod_{i=1}^{d}\lfloor\log_{p_i}N\rfloor(p_i-1)=O((\log N)^d)$.

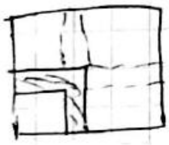
From this we get that for such I, $D((x_n)_{n=0}^{N-1};I)=O((\log N)^2)$.

Step 3 Take $\mathcal{S}'=\{[0,\frac{b_1}{2^{m_1}})\times[0,\frac{b_2}{3^{m_2}})\mid \begin{smallmatrix}b_1\in\{0,\dots,2^{m_1}-1\}\\ b_2\in\{0,\dots,3^{m_2}-1\}\end{smallmatrix}\}$, $\mathcal{S}=\{[0,b_1)\times[0,b_2)\mid b_1,b_2\in[0,1]\}$.

For each $I=[0,b_1)\times[0,b_2)\in\mathcal{S}$, take b_1',b_2' s.t. $\frac{b_1'}{2^{m_1}}\le b_1<\frac{b_1'+1}{2^{m_1}}$, $\frac{b_2'}{3^{m_2}}\le b_2<\frac{b_2'+1}{3^{m_2}}$,

and $I_1=[0,\frac{b_1'}{2^{m_1}})\times[0,\frac{b_2'}{3^{m_2}})$, $I_2=[0,\frac{b_1'+1}{2^{m_1}})\times[0,\frac{b_2'+1}{3^{m_2}})$.

So $I_1\subset I\subset I_2$ and $Vol(I_2\setminus I_1)\le\left(\frac{1}{2^{m_1}}+\frac{1}{3^{m_2}}\right)$



So $Vol(I_2\setminus I_1)N=O(1)$ and we finish by the Prop.

#5
28.11.16

Proof that quadratic irrationals are BA: Take a quadratic irrational $\alpha\notin\mathbb{Q}$.
There are $a,b,c\in\mathbb{Z}$ s.t. $a\alpha^2+b\alpha+c=0$. Write $ax^2+bx+c=a(x-\alpha)(x-\beta)$.
For any p,q s.t. $|\alpha-\frac{p}{q}|<|\beta-\frac{p}{q}|$: $|a(\frac{p}{q})^2+b(\frac{p}{q})+c|\ge\frac{1}{q^2}$ since it is not 0.
But $|a(\frac{p}{q})^2+b(\frac{p}{q})+c|=|a|\,|\frac{p}{q}-\alpha|\,|\frac{p}{q}-\beta|\le\frac{a}{2}|\alpha-\beta|\,|\frac{p}{q}-\alpha|$. Thus $|\frac{p}{q}-\alpha|\ge\frac{c}{q^2}$. $\blacksquare$

Two directions to extend the results about $(n\alpha)_{n=0}^{\infty}$ to higher dimensions
We can take $\underline{\alpha}\in\mathbb{R}^d$ and take $(n\underline{\alpha})_{n=0}^{\infty}\subset U^d=[0,1)^d=\mathbb{T}^d=\mathbb{R}^d/\mathbb{Z}^d$, i.e. take $\alpha=\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_d\end{pmatrix}$
and $n\underline{\alpha}=\begin{pmatrix}n\alpha_1 \bmod \mathbb{Z}\\ \vdots\\ n\alpha_d \bmod\mathbb{Z}\end{pmatrix}$.

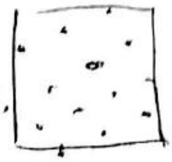
We will show (Schmidt '64): For a.e. $\underline{\alpha}\in\mathbb{R}^d$ $\forall\varepsilon>0$ and any N, $D((n\alpha)_{n=0}^{N-1};R_d)=O((\log N)^{d+1+\varepsilon})$
(we'll use Erdős-Turán-Koksma).
There is a result of Beck ('94): For a.e. $\underline{\alpha}\in\mathbb{R}^d$, $\forall\varepsilon>0,\forall N$ $D((n\alpha)_{n=0}^{N-1};R_d)=O((\log N)^{d+\varepsilon})$.
(This is hard and we won't prove it).
Remarks 1. Beck actually showed $O((\log N)^d(\log\log N)^{1+\varepsilon})$.
2. It is unknown whether there are $\underline{\alpha}$ s.t. $D((n\underline{\alpha})_{n=0}^{N-1};R_d)=O((\log N)^d)$ (in $d\ge2$).
Littlewood Conjecture $\forall\alpha=\binom{\alpha_1}{\alpha_2}\in\mathbb{R}^2$, $\liminf_{n\to\infty} n\langle n\alpha_1\rangle\langle n\alpha_2\rangle=0$.
If it is false, i.e. there is $\alpha=\binom{\alpha_1}{\alpha_2}$ s.t. $\liminf>0$, then α would satisfy $D((n\alpha);R_2)=O((\log N)^2)$
3. No explicit values of $\underline{\alpha}$ are known (for $d\ge2$) that satisfy the conclusion of Beck's theorem.

Second direction of generalization:

Suppose we use $(n\alpha)_{n=0}^{N-1}$ to solve static discrepancy problem in $d=1$. We get $(n\alpha, \frac{n}{N})_{n=0}^{N-1}$ as a low discrepancy sequence in $\mathcal{U}^2$. Here $\underset{\in\mathbb{T}^1}{n\alpha} = \underset{\in\mathbb{R}}{n\alpha - p}$, $p = \lfloor n\alpha \rfloor$. We are looking at $\binom{n\alpha-p}{\frac{n}{N}} \in \mathcal{U}^2$.

$\left\{\binom{n\alpha-p}{\frac{n}{N}}\right\} \cap \mathcal{U}^2 = \left\{ p\binom{1}{0} + n\binom{\alpha}{\frac{1}{N}} \mid p,n\in\mathbb{Z}\right\} \cap \mathcal{U}^2 = \left(\mathbb{Z}\binom{1}{0} + \mathbb{Z}\binom{\alpha}{\frac{1}{N}}\right)\cap\mathcal{U}^2.$

This means that we are looking at $\Lambda\cap\mathcal{U}^2$, $\Lambda\subset\mathbb{R}^2$ is a lattice (will be defined soon).



Note that $\left[\mathbb{Z}\binom{1}{0} + \mathbb{Z}\binom{\alpha}{\frac{1}{N}}\right] = \begin{pmatrix}1&0\\0&\frac{1}{N}\end{pmatrix}\left[\mathbb{Z}\binom{1}{0}+\mathbb{Z}\binom{\alpha}{1}\right]$.

General Construction Consider a lattice $\Lambda\subset\mathbb{R}^d$. For every N, take a diagonal matrix $T = T(N)$ and look at the discrepancy of $\mathcal{U}^d\cap T(\Lambda)$ (*).

Results For certain lattices ("admissible") Λ, the sets constructed in (*) satisfy

$D((x_n)_{n=0}^{N-1}; R_d) = O((\log N)^{d-1})$. We will prove this (Skriganov '94).

There are explicit admissible lattice constructed from algebraic number fields of degree d.

Regarding a.e. behavior, Skriganov ('98) proved: For every lattice Λ', for every $\varepsilon>0$, for a.e. $\mathcal{O}$ an orthogonal matrix, for every T a diagonal matrix of det $\frac{1}{N}$, set $\Lambda=\mathcal{O}(\Lambda')$. Then $D((*); R_d) = O((\log N)^{d-1+\varepsilon})$. (We will not prove this).

Thm 1 (Schmidt '64) For any $d\ge 1$, for a.e. $\alpha\in\mathbb{R}^d$, for any $\varepsilon>0$, $D((n\alpha)_{n=0}^{N-1}; R_d) = O(\log N^{d+1+\varepsilon})$

(The constant in the big O notation can depend on d,α,ε, not on N).

Recall the theorem of Erdős-Turán-Koksma: Let $(x_n)_{n=0}^{N-1}\subseteq\mathcal{U}^d$, k a paramater, then

$$D((x_n)_{n=0}^{N-1}; R_d) = O\left(\frac{N}{k} + \sum_{\substack{0<\|h\|_\infty\le k\\ h\in\mathbb{Z}^d}} \frac{1}{r(h)}\left|\sum_{n=0}^{N-1} e(hx_n)\right|\right) \text{ where } e(hx) = e^{2\pi i\langle h,x\rangle},\ r(h) = \prod_{i=1}^{d} \max\{1,|h_i|\}.$$

(We proved the case $d=1$, we will not prove the general case but the idea is similar).

We will apply this using $x_n = n\alpha$, $k=N$, so we are trying to ~~get a~~ bound $\sum_{0<\|h\|_\infty\le k}\frac{1}{r(h)}\left|\sum_{n=0}^{N-1} e(hn\alpha)\right| = O(\log N^{d+1+\varepsilon})$.

$\left|\sum_{n=0}^{N-1} e(hn\alpha)\right| = \left|\sum_{n=0}^{N-1} e(h\alpha)^n\right| = \left|\frac{1-e(hN\alpha)}{1-e(h\alpha)}\right| \le \frac{2}{|1-e(h\alpha)|} \le \frac{2c}{\|\langle h,\alpha\rangle\|}$ where $\|x\| = \langle x\rangle = \mathrm{dist}(x,\mathbb{Z})$ (confusing notation)

Thus, in order to prove Thm 1, it suffices to show Thm 2 (Schmidt '64): For a.e. $\alpha\in\mathbb{R}^d$, for any k, any $\varepsilon>0$,

$$\sum_{\substack{0<\|h\|_\infty\le k\\ h\in\mathbb{Z}^d}} \frac{1}{r(h)}\frac{1}{\|\langle h,\alpha\rangle\|} = O(\log(k)^{1+d+\varepsilon}).$$

Preparations for the proof of Thm 2

Borel-Cantelli Lemma $(X,\mathcal{B},\mu)$ a measure space, $\mu(X)<\infty$. $A_1,A_2,\ldots\in\mathcal{B}$. Define $\bar{A} = \{x\in X \mid x\in A_n \text{ for infinitely many } n\} = \{x\in X\mid \forall n\ \exists m\ge n\ \ x\in A_m\} = \bigcap_{n=1}^{\infty}\bigcup_{m\ge n} A_m =: \limsup A_n$. Suppose $\sum\mu(A_n)<\infty$. Then $\mu(\bar{A})=0$.

Proof Let $\varepsilon>0$. choose n s.t. $\sum_{m\ge n}\mu(A_m) < \varepsilon$. $\bar{A}\subseteq\bigcup_{m\ge n}A_m$ thus $\mu(\bar{A})\le\sum_{m\ge n}\mu(A_m)<\varepsilon$. Thus $\mu(\bar{A})=0$. ∎

Examples of Applications of Borel-Cantelli

Def: If $\exists\, \varepsilon>0$ and infinitely many p,q s.t. $|x-\frac{p}{q}| < \frac{1}{q^{2+\varepsilon}}$ then x is called very well-approximable (VWA).

Prop VWA has Lebesgue measure 0.

Proof: VWA is invariant under adding integers, so it is enough to show that $\mu(VWA\cap\mathcal{U}^1)=0$.

It's enough to prove that $VWA^{(\varepsilon)}=\{x\in[0,1]\mid \exists$ inf. many p,q satisfying $|x-\frac{p}{q}|<\frac{1}{q^{2+\varepsilon}}\}$ has measure zero, because $VWA\cap[0,1]=\bigcup_{\varepsilon>0}VWA^{(\varepsilon)}=\bigcup_k VWA^{(\frac{1}{k})}$.
The sequence is increasing

$|x-\frac{p}{q}|<\frac{1}{q^{2+\varepsilon}} \iff x\in\left(\frac{p}{q}-\frac{1}{q^{2+\varepsilon}},\frac{p}{q}+\frac{1}{q^{2+\varepsilon}}\right)$. Denote $A_{p,q}=\left(\frac{p}{q}-\frac{1}{q^{2+\varepsilon}},\frac{p}{q}+\frac{1}{q^{2+\varepsilon}}\right)$.

By definition, $VWA^{(\varepsilon)}=\limsup\,(A_{p,q})_{\substack{q=1\\ -q\le p\le 2q}}^{\infty}$. $\sum_{p,q}\mu(A_{p,q})=\sum_{q=1}^{\infty}\sum_{p=-q}^{2q}\mu(A_{p,q})=$

$$=\sum_{q=1}^{\infty}3q\frac{2}{q^{2+\varepsilon}}=6\sum_{q=1}^{\infty}\frac{1}{q^{1+\varepsilon}}<\infty.$$ Thus by Borel-Cantelli, $\mu(VWA^{(\varepsilon)})=0$. ∎

Prop For a.e. $\alpha\in\mathbb{R}^d$ there are only finitely many $h\in\mathbb{Z}^d$, $h_i\neq0$ s.t. $\|\langle h,\alpha\rangle\|\le\frac{1}{(h_1\cdots h_d)^2}$ (*)
(as we will see, 2 can be replaced with $1+\delta$, $\delta>0$).

Proof: As before, it suffices to prove for a.e. $\alpha\in\mathcal{U}^d$. For each $h\in\mathbb{Z}^d$, define $A_h=\{\alpha\in\mathcal{U}^d,\ (*)\text{ holds for }\alpha,h\}$. By Borel-Cantelli, it's enough to show that $\sum\mu(A_h)<\infty$. Denote $h=(h_1,\dots,h_d)$. By partitioning the sum into sums on which signs of h_i are fixed, we can assume all $h_i>0$. $A_h=\{\alpha\in\mathcal{U}^d\mid \|\langle h,\alpha\rangle\|\le\frac{1}{(h_1\cdots h_d)^2}\}=\{\alpha\in\mathcal{U}^d\mid \exists p\in\mathbb{Z},\ p\in[0,\|h\|_1],\text{ s.t. } |\langle h,\alpha\rangle-p|\le\frac{1}{(h_1\cdots h_d)^2}\}$

$\subseteq\left\{\alpha\in\mathcal{U}^d\mid \exists p\in\mathbb{Z}\cap[0,\|h\|_1)\ \ dist(\alpha,\{x\mid\langle x,h\rangle=p\})\le\frac{1}{\|h\|_2(h_1\cdots h_d)^2}\right\}$. Thus,

$Leb(A_h)=O\left(\frac{\|h\|_1}{(h_1\cdots h_d)^2\|h\|_2}\right)=O\left(\frac{1}{(h_1\cdots h_d)^2}\right)$. So we get that

$$\sum_{\substack{h_i>0\\ i=1,\dots,d}}\mu(A_h)=\sum_{h_1=1}^{\infty}\cdots\sum_{h_d=1}^{\infty}\frac{1}{(h_1\cdots h_d)^2}=\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^d<\infty.$$ ∎

Proof of Thm2: Set $\delta=\frac{\varepsilon}{d+1}$.

Step 1 Reduce to case that all $h_i\neq0$. (We'll postpone that for now).

~~(We'll apply)~~

~~Define $J(h)=\int_0^1 d\alpha_d\cdots\int_0^1 d\alpha_1[\log$~~

Define $J(h)=\int_0^1 d\alpha_d\cdots\int_0^1 d\alpha_1\left[\|\langle h,\alpha\rangle\|\log(\|\langle h,\alpha\rangle\|)^{1+\delta}\right]^{-1}$.

Step 2 For fixed δ, for all h, $J(h)$ converges and is bounded by a number independent of h.

We conclude the proof, assuming steps 1,2,3: WLOG, we can assume $h_i>0$ by partitioning into 2^d possibilities depending on the signs of the h_i.

Step 3 Reduce to the case $|h_i|>2$.

We're trying to bound for a.e. α, $\sum_{h_1=2}^{k}\cdots\sum_{h_d=2}^{k}\frac{1}{h_1\cdots h_d}\frac{1}{\|\langle h,\alpha\rangle\|}$.

Consider $\sum_{h_1=2}^{\infty}\cdots\sum_{h_d=2}^{\infty}\left[h_1\log(h_1)^{1+\delta}\cdots h_d\log(h_d)^{1+\delta}\right]^{-1}J(h)<\infty$.

$$\sum\cdots\sum[\cdots]J(h)\ge\int_{\alpha\in\mathcal{U}^d}\sum_{h_1,\dots,h_d=2}^{\infty}\left[h_1\log(h_1)^{1+\delta}\cdots h_d\log(h_d)^{1+\delta}\,\|\langle h,\alpha\rangle\|\log(\|\langle h,\alpha\rangle\|)^{1+\delta}\right]^{-1}d\alpha$$

Therefore for a.e. $\alpha \in \mathcal{U}^d$, (1) $\sum_{h_1=2}^{\infty} \dots \sum_{h_d=2}^{\infty} [h_1 \log(h_1)^{1+\delta} \dots h_d \log(h_d)^{1+\delta} \|\langle h, \alpha\rangle\| \log(\|\langle h, \alpha\rangle\|)^{1+\delta}]^{-1} < \infty$

(since the integral is finite). From the application of Borel-Cantelli, (2) $\exists c$ $\|\langle h, \alpha\rangle\| \geq \frac{c}{(h_1 \dots h_d)^2}$ for all h.

Let's look at the α for which (1),(2) hold.

$(2) \Rightarrow \left|\log(\|\langle h, \alpha\rangle\|)\right| \leq 2\log(h_1 \dots h_d) + \log c$ (since $\log(\|x\|) < \infty$).

$$\sum_{h_1,\dots,h_d=2}^{k} [h_1 \dots h_d \|\langle h, \alpha\rangle\|]^{-1} \leq \Big[\max_{\substack{2 \leq h_i \leq k \\ i=1,\dots,d}} \log(h_1)^{1+\delta} \dots \log(h_d)^{1+\delta} \cdot \log \|\langle h, \alpha\rangle\|^{1+\delta}\Big] \cdot$$

$$\cdot \underbrace{\sum_{h_1,\dots,h_d=2}^{\infty} [h_1 \log(h_1)^{1+\delta} \dots h_d \log(h_d)^{1+\delta} \|\langle h, \alpha\rangle\| \log \|\langle h, \alpha\rangle\|^{1+\delta}]^{-1}}_{\text{finite by } \infty} \leq \log(k)^{(1+\delta)d} \cdot c_1 \log(k^d)^{1+\delta} \cdot c_2 =$$

$$= O(\log k^{(1+\delta)(d+1)}) = O(\log(k)^{d+1+\varepsilon}). \quad \blacksquare$$

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5.12.16

Recall that our goal is to construct lattices Λ, diagonal matrices T with small determinant, and vectors x s.t. $\mathcal{U}^d \cap (T\Lambda + x)$ has small discrepancy.

<u>Notation</u> If $v_1, \dots, v_d \in \mathbb{R}^d$ linearly independent, $\Lambda = \mathbb{Z}v_1 + \dots + \mathbb{Z}v_d = \{\sum a_i v_i \mid a_i \in \mathbb{Z}\}$ is called a <u>lattice</u>. $v_1, \dots, v_d$ are called a <u>basis</u> of Λ.

Written differently, Let $A = ((v_1) \dots (v_d))$, then $\Lambda = A(\mathbb{Z}^d)$.

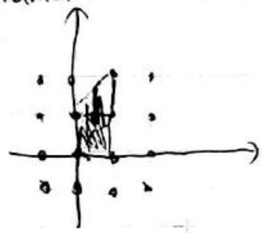
Note that $A \in SL_d(\mathbb{Z}) = \{A \in M_n(\mathbb{Z}), \det A = 1\} \iff A(\mathbb{Z}^d) = \mathbb{Z}^d$. If $v_1, \dots, v_d$ is a basis of the lattice, then any other lattice can be obtained by right-multiplication by a matrix in $SL_d(\mathbb{Z})$.

A <u>fundamental domain</u> of Λ is a measurable set P s.t. for any $\lambda_1, \lambda_2 \in \Lambda$, $(\lambda_1 + P) \cap (\lambda_2 + P) = \emptyset$ and $\bigcup_{\lambda \in \Lambda} (\lambda + P) = \mathbb{R}^d$.

<u>Example</u> 1. If $\Lambda = \mathbb{Z}^d$ then $P = \mathcal{U}^d$ is a fundamental domain.

2. If $\Lambda = A(\mathbb{Z}^d)$ then $A(\mathcal{U}^d)$ is a fundamental domain. $\mathrm{Vol}(A(\mathcal{U}^d)) = |\det(A)|$ is called the co-volume of Λ $\mathrm{covol}(\Lambda)$. Fact: all fund. domains have the same volume.

<u>Notation</u> If $v = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} \in \mathbb{R}^d$, $N_m(v) = \prod_{i=1}^{d} v_i$. If $X = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_d \end{pmatrix}$, $N_m(X) = \prod x_i$.



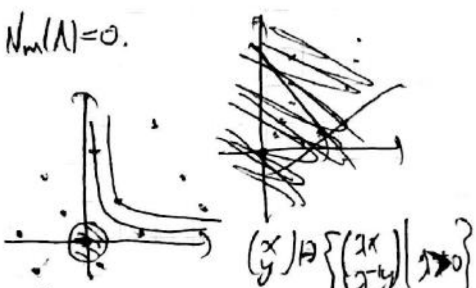
If $\Lambda \subset \mathbb{R}^d$ is a lattice, $N_m(\Lambda) = \inf_{v \in \Lambda \setminus \{0\}} |N_m(v)|$.

<u>Def</u> Λ is <u>admissible</u> if $N_m(\Lambda) > 0$.

<u>Examples</u> If Λ contains $x = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}$ with some $x_i = 0$, $x \neq 0$, then $N_m(\Lambda) = 0$.

If Λ contains x, y s.t. $x_i = \pm y_i$ for some i, $N_m(\Lambda) = 0$.

<u>Remark</u> For typical Λ, $N_m(\Lambda) = 0$, but Λ has no vector with a zero coefficient.



$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \left\{\begin{pmatrix} 2x \\ 2^{-1}y \end{pmatrix} \middle| \, 2 > 0\right\}$

if Λ is admissible, there's a neighborhood of 0 outside of the hyperbola

<u>Thm 1</u> Suppose Λ is an admissible lattice. Then there are constants c, N_0 (depending on Λ) s.t. for any invertible diagonal matrix T, and any $z \in \mathbb{R}^d$, if we set $\Omega_{T,z}(\Lambda) = (T\Lambda - z) \cap \mathcal{U}^d$,

and if $N = \#\mathcal{R}_{T,z}(N) \geq N_0$, then $D(\mathcal{R}_{T,z}(\Lambda); R_d^{\delta}) \leq c(\log(N))^{d-1}$.

Remark: By varying T continously we can find examples with arbitrary N.

More Notation: Let $\Lambda \subset \mathbb{R}^d$ be a translate of a lattice, $\mathcal{O} \subset \mathbb{R}^d$ compact, convex.

Let $R(\mathcal{O}, \Lambda) = \#(\mathcal{O} \cap \Lambda) - \frac{\mathrm{Vol}(\mathcal{O})}{\mathrm{covol}(\Lambda)}$, $r(\mathcal{O}, \Lambda) = \sup_{x \in \mathbb{R}^d} \# R(\mathcal{O}, \Lambda + x) = \sup_{x \in F} R(\mathcal{O}, \Lambda + x)$. for any fund. domain F

$\lambda_1(\Lambda) = \min_{v \in \Lambda \setminus \{0\}} \|v\|_2$.

Thm 2: Let $\Lambda \subset \mathbb{R}^d$ be an admissible lattice. Then there are c_1, N_1 depending only on $\lambda_1(V), N_m(\Lambda)$ and $\mathrm{covol}(\Lambda)$ s.t. for any invertible diagonal T, $r(T([-\frac{1}{2}, \frac{1}{2}]^d), \Lambda) < c_1 (\log N_m(T))^{d-1}$ whenever $N_m(T) \geq N_1$.

Background - Gauss circle problem: Fix $\Lambda = \mathbb{Z}^2$, $\mathcal{O} = B(0, T)$. $N_T = \#(\Lambda \cap B(0, T))$.

Gauss proved that $N_T = \pi T^2 + O(T)$. Gauss asked whether the error could be made smaller.

Conjecture $\forall \varepsilon > 0$ $N_T = \pi T^2 + O_\varepsilon(T^{\frac{1}{2} + \varepsilon})$.

Landau proved $N_T = \pi T^2 + \Omega(T^{\frac{1}{2}} (\log T)^{\frac{1}{4}})$, i.e. along a subsequence, $|N_T - \pi T^2| > c\, T^{\frac{1}{2}} (\log T)^{\frac{1}{4}}$.

Best result to date: Huxley (2003?): $N_T = \pi T^2 + O(T^{\frac{131}{208}})$.

Constructions of admissible lattices

Ex: In $\mathbb{R}^2$, $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbb{Z}^2$, $ad - bc \neq 0$, $a \neq 0 \neq c$, is admissible $\Leftrightarrow$ $\frac{b}{a}, \frac{d}{c}$ are both BA.

e.g. $\begin{pmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{3} \end{pmatrix} \mathbb{Z}^2$.

Constructing admissible lattices in $\mathbb{R}^d$, $d > 2$, is harder.

"Geometric embedding of ring of integers" or "Norm tori".

Let k be a totally real number field, i.e. $k \subseteq \mathbb{R}$ is a field, $[k : \mathbb{Q}] = d < \infty$, s.t. for any of the field embeddings $L : k \hookrightarrow \mathbb{C}$, $L(k) \subseteq \mathbb{R}$.

Recall that there d embeddings $k \hookrightarrow \mathbb{C}$, call them $\sigma_1, \ldots, \sigma_d$, $\sigma_1 = \mathrm{id}$.

Let $\mathcal{O}_k$ be the ring of algebraic integers of k (i.e. minimal polynomial is monic).

Prop If $\beta_1, \ldots, \beta_d \in \mathcal{O}_k$ generate k over $\mathbb{Q}$, $M = (\sigma_i(\beta_j))_{i,j = 1, \ldots, d}$, then $M(\mathbb{Z}^d)$ is admissible.

Example Suppose $f \in \mathbb{Z}[x]$, $\deg f = d$, f monic, all roots of f are real, and f is irreducible over $\mathbb{Q}$.

Let $\alpha = \alpha_1$ be a root, and $\alpha_2, \ldots, \alpha_d$ the other roots. Set $k = \mathbb{Q}(\alpha)$. The map $\alpha \mapsto \alpha_i$ extends to a field embedding $\sigma_i : k \to \mathbb{Q}(\alpha_i)$. Choose $\beta_i = \alpha^{i-1}$, $M = \begin{pmatrix} 1 & \alpha_1 & \ldots & \alpha_1^{d-1} \\ \vdots & & & \\ 1 & \alpha_d & \ldots & \alpha_d^{d-1} \end{pmatrix}$.

Proof M is invertible (exercise). This implies $M(\mathbb{Z}^d)$ is a lattice. To show admissibility,

Let $v = M \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$, $p_i \in \mathbb{Z}$ not all zero. $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $v_i = \sum p_j \sigma_i(\beta_j) = \sigma_i(\underbrace{\sum p_j \beta_j}_{x \in k \setminus \{0\}}) \in k \setminus \{0\}$.

$N_m(v) = v_1 \ldots v_d = \sigma_1(x) \ldots \sigma_d(x)$. Claim $|N_m(v)| \geq 1$. To get this, we'll show $N_m(v)$ is an integer in $\mathbb{Q}$.

Each $\sigma_i(x)$ is an algebraic integer, thus $N_m(x)$ is an algebraic integer.

If $\sigma:\mathbb{C}\to\mathbb{C}$ is any field automorphism, $\sigma(N_m(v)) = \prod \sigma\circ\sigma_i(\alpha) = \prod \sigma_i(\alpha) = N_m(v)$.

Thus $N_m(v)\in\mathbb{Q}$. Since it is an algebraic integer, $N_m(v)\in\mathbb{Z}$. Q.E.D.

Conjecture (Cassels–Swinnerton-Dyer '55) For $d>2$, the only admissible lattices arise from the preceding construction, up to the action of the diagonal group $\bar{A}=\left\{\begin{pmatrix}a_1&&0\\&\ddots&\\0&&a_d\end{pmatrix} \;\middle|\; \prod a_i\neq 0,\ a_i>0\right\}$.

They proved that this conjecture implies Littlewood's conjecture.

Let $A=\{a\in\bar{A} \mid \det a=1\}$. $A\cong\mathbb{R}^{d-1}$ as Lie groups.

Prop Λ is admissible $\iff \inf_{a\in A}\lambda_1(a\Lambda)>0$.

Proof ($\Rightarrow$) Set $\eta=\inf_{v\in\Lambda\setminus\{0\}}|N_m(v)|>0$. Let $a\in A$, $u\in a\Lambda\neq 0$. $u_i=a_iv_i$ where $v=\begin{pmatrix}v_1\\\vdots\\v_n\end{pmatrix}$, $u=av$, $a=\begin{pmatrix}a_1&&\\&\ddots&\\&&a_n\end{pmatrix}$. $u=\begin{pmatrix}u_1\\\vdots\\u_n\end{pmatrix}$.

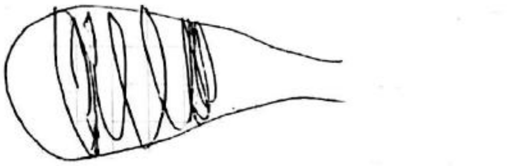
$$\|u\|_2^2=\sum a_i^2v_i^2 \underset{\text{AM-GM}}{\geq} d\prod(a_i^2v_i^2)^{\frac{1}{d}} = d\left(\prod v_i^2\right)^{\frac{1}{d}}\cdot 1 = d\,\eta^{\frac{2}{d}}>0 \Rightarrow \blacksquare$$

($\Leftarrow$) ex.

Remark There is a space of lattices, $SL_d(\mathbb{R})/SL_d(\mathbb{Z}) = \{\text{lattices of covol } 1\}$. A acts on this space: $a\cdot\Lambda\mapsto a\Lambda$. Mahler compactness criterion: A subset X of the space of lattices has compact closure $\iff \inf_{x\in X}\lambda_1(x)>0$. Thus the proposition says that Λ is admissible $\iff$ the orbit $A\Lambda$ has compact closure.

Prop (will not be proved, uses Dirichlet's theorem on units): Λ constructed as in previous question $\Rightarrow A\Lambda$ is compact.

So a reformulation of Cassels–S.D. conj. is: Any A-orbit which has compact closure is actually compact for $d>2$.



The covering radius of a lattice

Suppose $\Lambda\subset\mathbb{R}^d$ is a lattice.

$$\mathrm{covrad}(\Lambda)=\sup_{y\in\mathbb{R}^d}\inf_{x\in\Lambda}\|x-y\|_2 = \inf\{r>0 \mid \mathbb{R}^d=\bigcup_{\lambda\in\Lambda}B(\lambda,r)\} = \mathrm{diam}(\mathbb{R}^d/\Lambda)$$ (w.r.t. Euclidean metric on the torus $\mathbb{R}^d/\Lambda$).

~~Suppose~~

Prop (1) Suppose $r=\mathrm{covrad}(\Lambda)$. Then $\overline{B(0,r)}$ contains a fundamental domain.

(2) for any d There are two functions $f_1,f_2:(0,\infty)\to(0,\infty)$ s.t. $f_1(\delta)<f_2(\delta)\ \forall\delta$ and $f_i(\delta)\xrightarrow[\delta\to 0+]{}\infty$, and for any lattice $\Lambda\subset\mathbb{R}^d$ of covolume 1, $f_1(\lambda_d(\Lambda))\leq\mathrm{covrad}(\Lambda)\leq f_2(\lambda_1(\Lambda))$.

Proof $\mathrm{Vor}(\Lambda)$ = Voronoi cell of λ $=\{x\in\mathbb{R}^d \mid \|x\|_2=\min_{\lambda\in\Lambda}\|x-\lambda\|_2\}$. Clearly $\bigcup_{\lambda\in\Lambda}\mathrm{Vor}+\lambda=\mathbb{R}^d$.
a convex polytope

By "removing some of the boundary", one can find a fundamental domain contained in Vor.


$\mathrm{Vor}(\Lambda)$.

But obviously $\mathrm{Vor}\subseteq\overline{B(0,r)}$. This proves (1).

Idea of proof of (2) For RHS, given δ, one needs to find $r = f_2(\delta)$ s.t. any lattice Λ with $\lambda_1(\Lambda) > \delta$ has $\mathrm{covrad}(\Lambda) \le r$. By def. of Vor, $B(0, \frac{\delta}{2}) \subset \mathrm{Vor}$: If $y \in B(0, \frac{\delta}{2})$, $y \notin \mathrm{Vor}$, there is $\lambda \in \Lambda$ with $\|\lambda - y\|_2 \le \|y\|_2 < \frac{\delta}{2}$, thus $\|\lambda\|_2 \le \|y\|_2 + \|\lambda - y\|_2 < \delta < \lambda_1(\Lambda)$, a contradiction!

Let T be the largest length of a point $V_{max} \in \mathrm{Vor}$.

Consider $\mathrm{Conv}(B(0, \frac{\delta}{2}) \cup V_{max})$. This contains a pyramid with base a ball of radius $\frac{\delta}{2}$ and dim $d-1$ and height T.

Note that $T = \mathrm{covrad}(\Lambda)$. So $1 \ge \mathrm{Vol}(\mathrm{Vor}) \ge \mathrm{Vol}(P) \ge c\,\delta^{d-1}\, T \xrightarrow[T \to \infty]{} \infty$. So $T \le f_2(\delta)$. ∎

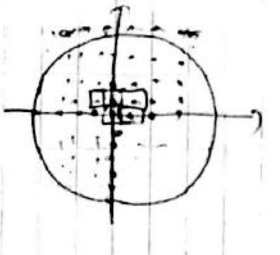
Prop For any lattice Λ and any $\mathcal{O} \subset \mathbb{R}^d$ compact, convex, with nonempty interior. Then $\exists t_0, c\ \forall t \ge t_0\ \forall x \in \mathbb{R}^n$ $\left| \#(\Lambda \cap (t\mathcal{O} + x)) - \frac{\mathrm{Vol}(\mathcal{O})\, t^d}{\mathrm{Covol}(\Lambda)} \right| < c\, t^{d-1}$. The constants c, t_0 depend only on $\mathcal{O}$, $\mathrm{covol}(\Lambda)$, $\lambda_1(\Lambda)$.

In particular $\exists t_0$ s.t. $\forall t > t_0$, for all lattices of covolume 1 with a fixed lower bound on $\lambda_1(\Lambda)$,

$$\frac{1}{2} \#([-t,t]^d \cap \Lambda) \le 2^d t^d \le 2\#([-t,t]^d \cap \Lambda).$$

Remark this gives Gauss' easy estimate in the circle problem.

Picture for $d=2$, $\mathcal{O} = B(0,t)$:



Let $A = \bigcup_{\lambda + \mathrm{Vor} \subset B(0,t)} \lambda + \mathrm{Vor}$

$B = \bigcup_{(\lambda + \mathrm{Vor}) \cap B(0,t) \neq \emptyset} \lambda + \mathrm{Vor}$

$A \subseteq B(0,t) \subset B$.

Then $\mathrm{Vol}(A) \ge \pi(t-2)^2$, $\mathrm{Vol}(B) \le (t+2)^2\pi$. We can see that

$$\pi t^2 - \mathrm{Vol}(A) \le \#B(0,t) \cap \Lambda - \pi t^2 \le \mathrm{Vol}(B) - \pi t^2$$
$$-4\pi t + 4\pi \le |\#B(0,t) \cap \Lambda - \pi t^2| \le 4\pi t + 4\pi \quad \Rightarrow \quad |\#B(0,t) \cap \Lambda - \pi t^2| = O(t).$$

#7
12.12.16

Prop (trivial bound in Gauss circle problem) (again...): For any lattice $\Lambda \subseteq \mathbb{R}^n$, $\mathrm{Covol}(\Lambda) = 1$, and any $x \in \mathbb{R}^d$, any $T > 0$, any convex $\mathcal{O} \subseteq \mathbb{R}^d$, bounded with non-empty interior, $\#(T\mathcal{O} + x) \cap \Lambda = \mathrm{Vol}(\mathcal{O})\, T^d + O(T^{d-1})$ with implicit constants depending only on $\mathcal{O}$, $\lambda_1(\Lambda)$.

Proof WLOG assume $0 \in \mathrm{int}(\mathcal{O})$. Let F be a fundamental domain for Λ, so $\mathrm{vol}(F) = 1$, and we can choose F s.t. $r = \mathrm{diam}(F)$ is bounded from above by a number depending only on $\lambda_1(\Lambda)$. (We saw that earlier). Let $\Omega = T\mathcal{O} + x$, $A = \{v \in \Lambda \mid v + F \subset \Omega\}$, $B = \{v \in \Lambda \mid (v+F) \cap \Omega \neq \emptyset\}$. There is a $c > 0$ (depending on $\mathcal{O}$) s.t. $(T+cr)\mathcal{O} + x$ contains an r-neighborhood of Ω, $(T-cr)\mathcal{O} + x$ is contained in $\bigcup_{v \in A} v + F$. $\#B = \mathrm{Vol}\left(\bigcup_{v \in B} (v+F)\right) \le \mathrm{Vol}((T+cr)\mathcal{O} + x) \le (1 + \frac{cr}{T})^d T^d \mathrm{vol}(\mathcal{O})$ and similarly $\#A \ge (1 - \frac{cr}{T})^d \mathrm{Vol}(\mathcal{O})\, T^d$

Thus $(1-\frac{cf}{T})^d \operatorname{Vol}(\mathcal{O})T^d \le \#A = \#\Lambda\cap\Omega \le \#B \le (1+\frac{cf}{T})\operatorname{Vol}(\mathcal{O})T^d \Rightarrow$ Q.E.D.

Corollary. If $\Lambda\subset\mathbb{R}^d$ is admissible, $\operatorname{covol}(\Lambda)=1$, then $\exists T_0>0$ s.t. for any box $B=Y(\mathcal{U}^d)$ where Y is diagonal, with $|N_m(Y)|\ge T_0$ and any $z\in\mathbb{R}^d$, $\frac{1}{2}\#B\cap(\Lambda+z)\le|N_m(Y)|\le 2\#(B\cap(\Lambda+z))$

Proof: Note that if $Y=\begin{pmatrix}t&&\\&\ddots&\\&&t\end{pmatrix}$, $t^d=N_mY=T\ge T_0$, Then $B=t\mathcal{U}^d$ and the previous Prop. can be used (with $\mathcal{O}=\mathcal{U}^d$). For general $Y=\begin{pmatrix}y_1&&\\&\ddots&\\&&y_d\end{pmatrix}$, wlog $y_i>0$ $\forall i$, $T=N_m(Y)$, $t=T^{\frac{1}{d}}$. We can write $Y=\begin{pmatrix}t&&\\&\ddots&\\&&t\end{pmatrix}\begin{pmatrix}a_1&&\\&\ddots&\\&&a_d\end{pmatrix}$, $a_i=\frac{y_i}{t}$, $a=\begin{pmatrix}a_1&&\\&\ddots&\\&&a_d\end{pmatrix}\in A=\{a\mid\det a=1\}$. Then $\#B\cap(\Lambda+z)=\#a\begin{pmatrix}t&&\\&\ddots&\\&&t\end{pmatrix}\mathcal{U}^d\cap(\Lambda+z)=\#[0,t)^d\cap(a^{-1}(\Lambda)+a^{-1}(z))$. Recall that Λ is admissible iff $\inf_{a\in A}\lambda_1(a\Lambda)>0$, so $\lambda_1(a^{-1}\Lambda)$ is bounded below by a constant independent of a. Thus by the previous prop. we get a bound $\frac{1}{2}\#B\cap(\Lambda+z)\le\operatorname{Vol}(B)\le 2\#B\cap(\Lambda+z)$ for $\operatorname{Vol}(B)\ge T_0$, T_0 independent on a. Q.E.D.

Our goal is Thm 1: Let $\Lambda\subset\mathbb{R}^d$ be an admissible lattice. Then there are c_0, N_0 (depending on Λ) s.t. for any invertible diagonal Y, and $z\in\mathbb{R}^d$, If we denote $\Omega_{Y,z}(\Lambda)=\mathcal{U}^d\cap Y^{-1}(\Lambda-z)$, $N=\#\Omega_{Y,z}(\Lambda)$, and $N\ge N_0$, then $|D(\Omega_{Y,z}(\Lambda;R_J)|\le c\log(N)^{d-1}$.

Notation: For $\Lambda\subset\mathbb{R}^d$, $\mathcal{O}\subset\mathbb{R}^d$ convex bounded, $R(\mathcal{O},\Lambda)=\#\mathcal{O}\cap\Lambda-\frac{\operatorname{Vol}(\mathcal{O})}{\operatorname{Covol}(\Lambda)}$, $r(\mathcal{O},\Lambda)=\sup_{x\in\mathbb{R}^d}|R(\mathcal{O}+x,\Lambda)|$.

Thm 2: If $\Lambda\subset\mathbb{R}^d$ is admissible, then $\exists c_1$ (depending only on $N_m(\Lambda)$, $\operatorname{covol}(\Lambda)$) s.t. for any invertible diagonal Y, $r(Y[-\frac{1}{2},\frac{1}{2}]^d,\Lambda)\le c_1(\log(2+|N_m(Y)|)^{d-1})$.

Thm 2 $\Rightarrow$ Thm 1: clearly for any $z\in\mathbb{R}^d$, invertible Y, $\mathcal{O},\Lambda$: $\#(\mathcal{O}+x\cap\Lambda)=\#(\mathcal{O}\cap\Lambda-x)$, $\#(Y\mathcal{O}\cap\Lambda)=\#(\mathcal{O}\cap Y^{-1}(\Lambda))$. Similarly for R,r. Let Ω,N be as in Thm 1. Recall that in order to bound disc. w.r.t $R_J=\{\text{axis parallel boxes}\}$ it suffices to bound w.r.t $R_J^*=\left\{\begin{array}{l}\text{axis-parallel}\\\text{boxes with corners}\\\text{at }0\end{array}\right\}=\left\{Y_0(\mathcal{U}^d)\;\middle|\;\begin{array}{l}Y_0=\begin{pmatrix}y_1&&\\&\ddots&\\&&y_d\end{pmatrix}\\0<y_i<1\end{array}\right\}$

$$\#(Y_0\mathcal{U}^d\cap\Omega_{Y,z}(\Lambda))-N\operatorname{Vol}(Y_0\mathcal{U}^d)=\#(Y_0\mathcal{U}^d\cap\Omega_{Y,z}(\Lambda))-N\cdot N_m(Y_0)=\#(Y_0\mathcal{U}^d\cap Y^{-1}(\Lambda-z))-N\,N_m(Y_0)$$
$$=\#(Y_0\mathcal{U}^d\cap Y^{-1}(\Lambda-z))-N_m(Y_0)\#((Y\mathcal{U}^d+z)\cap\Lambda)=\#((Y Y_0\mathcal{U}^d+z)\cap\Lambda)-N_m(Y_0)\#(Y\mathcal{U}^d+z\cap\Lambda)=$$
$$\underset{\substack{\text{add and subtract}\\N_m(Y)N_m(Y_0)}}{=}R(Y_0Y\mathcal{U}^d+z,\Lambda)-N_m(Y_0)R(Y\mathcal{U}^d+z,\Lambda).$$
So $D(\Omega_{Y,z}(\Lambda);R_J^*)\le|R(Y_0Y\mathcal{U}^d+z,\Lambda)|+|R(Y\mathcal{U}^d+z,\Lambda)|$

$$\le r(Y_0Y\mathcal{U}^d+z,\Lambda)+r(Y\mathcal{U}^d+z,\Lambda)\underset{\text{Thm 2}}{\le}c_2(\log(2+|N_m(Y_0)N_m(Y)|)^{d-1}+c_2(\log(2+N_m(Y)))^{d-1}$$
$$\underset{\substack{\text{by previous corollary, } N_m(Y)\approx N \\ \text{(up to mult. constant)}}}{\le}c_3\log(N)^{d-1}\quad(c_3\text{ depends on }c_2,\,2,\text{ and on const. from cor.})$$

$\boxed{}$

To prove Thm 2, we'll need some Fourier analysis on $\mathcal{U}^d\simeq\mathbb{R}^d/\mathbb{Z}^d$. For $h\in\mathbb{Z}^d$, set $e(h,x)=e^{2\pi i\langle h,x\rangle}$. It is well-defined on $\mathcal{U}^d$. To any $f:\mathcal{U}^d\to\mathbb{C}$ measurable, its Fourier series is $\sum_{h\in\mathbb{Z}^d}\hat{f}(h)e(h,x)$, $\hat{f}(h):=\int_{\mathcal{U}^d}f(x)e(-h,x)\,dx$

If $f\in L^2(\mathbb{T}^d)$ then (*) $f(x)=\sum_{h\in\mathbb{Z}^d}\hat f(h)e(h\cdot x)$ as functions in L^2, and $f\mapsto(\hat f(h))_{h\in\mathbb{Z}^d}$ is an isometry between $L^2(\mathbb{T}^d)$ and $\ell^2(\mathbb{Z}^d)$. We have pointwise convergence everywhere (and uniformly in x) when $\sum_{h\in\mathbb{Z}^d}|\hat f(h)|<\infty$ (by the Weierstrass M-test). If f is continuous and $\sum|\hat f(h)|<\infty$, (*) holds pointwise.

If f is a trig. polynomial, the RHS of (*) is finite and we also have good convergence.

If f is smooth, then $\forall A>0$ $\hat f(h)=O\left((1+\|h\|)^{-A}\right)$, constants depending on A, f.

Now suppose $\Lambda=A(\mathbb{Z}^d)$, A invertible, is a lattice in $\mathbb{R}^d$. A map $\mathbb{R}^d\xrightarrow{x\mapsto Ax}\mathbb{R}^d$ extends to $\mathbb{R}^d/\mathbb{Z}^d\to\mathbb{R}^d/\Lambda$.

By a change of variables formula, for $f:\mathbb{R}^d/\Lambda\to\mathbb{C}$, $f(z)=\frac{1}{\mathrm{covol}(\Lambda)}\sum_{h\in\Lambda^*}\hat f(h)e(h\cdot z)$, where Λ^* is the dual lattice of Λ, i.e. $\Lambda^*=(A^t)^{-1}(\mathbb{Z}^d)\underset{\text{Ex.}}{=}\{y\in\mathbb{R}^d\mid\forall x\in\Lambda,\langle y,x\rangle\in\mathbb{Z}\}$.

Poisson Summation Formula (for $\mathbb{Z}^d$). Suppose $f:\mathbb{R}^d\to\mathbb{C}$ measurable with compact support.

Then $\sum_{v\in\mathbb{Z}^d}f(v+x)=\sum_{h\in\mathbb{Z}^d}\hat f(h)e(h\cdot x)$ where $\hat f(h)=\int_{\mathbb{R}^d}f(x)e(-h\cdot x)dx$. Caution: Not the same $\hat f$ as before!

This equality holds for all x if f is C^∞. Idea of Proof: Define $F(x)=\sum_{v\in\mathbb{Z}^d}f(x+v)$. F is actually a function on $\mathbb{R}^d/\mathbb{Z}^d$, and if f is smooth so is F. Then by usual Fourier series for F we finish.

For general Λ, $\sum_{v\in\Lambda}f(v+x)=\frac{1}{\mathrm{covol}(\Lambda)}\sum_{h\in\Lambda^*}\hat f(h)e(h\cdot x)$.

We'll apply that to $f(x)=R(\mathcal{O}+x,\Lambda)=\#(\mathcal{O}+x\cap\Lambda)-\frac{\mathrm{Vol}(\mathcal{O})}{\mathrm{covol}(\mathcal{O})}$.

~~Prop: Suppose covol(Λ)=1. Then~~ $\cancel{R(\mathcal{O}+x,\Lambda)=\sum_{h\in\Lambda^*\setminus\{0\}}\hat\chi_{\mathcal{O}}(h)e(h\cdot x)}$ ~~where~~ $\cancel{\hat\chi_{\mathcal{O}}(h)=\int_{\mathbb{R}}}$

(Voronoi-Hardy Formula)

Prop: Suppose $\mathrm{covol}(\Lambda)=1$. Then $R(\mathcal{O}-x,\Lambda)\overset{L^2}{=}\sum_{h\in\Lambda^*\setminus\{0\}}\hat\chi_{\mathcal{O}}(h)e(h\cdot x)$, where $\hat\chi_{\mathcal{O}}(h)=\int_{\mathbb{R}^d}\chi_{\mathcal{O}}(x)e(-h\cdot x)\,dx=\int_{\mathcal{O}}e(-h\cdot x)\,dx$, where "$\overset{L^2}{=}$" is in L^2.

Proof: $R(\mathcal{O}-x,\Lambda)=\#(\mathcal{O}-x\cap\Lambda)-\mathrm{Vol}(\mathcal{O})$

$$\#(\mathcal{O}-x\cap\Lambda)=\sum_{v\in\Lambda}\chi_{\mathcal{O}-x}(v)=\sum_{v\in\Lambda}\chi_{\mathcal{O}}(x+v)\underset{PSF}{=}\sum_{h\in\Lambda^*}\hat\chi_{\mathcal{O}}(h).$$ Note that $\hat\chi_{\mathcal{O}}(0)=\mathrm{Vol}(\mathcal{O})$. ∎

Strategy: Bound $R(\mathcal{O}-x,\Lambda)$ from above by bounding RHS in the prop.

Problem: Prop is not pointwise equality because $\chi_{\mathcal{O}}$ is not smooth. We will fix by "smoothing" which we'll discuss soon.

Let's expand RHS of prop, for $\mathcal{O}=[-t,t]^d$. Assume Λ is admissible. For $h=(h_1,\dots,h_d)$, $\forall j\ h_j\neq0$,

$$\int_{\mathcal{O}}e(-h\cdot x)dx=\int_{-t}^{t}dx_1\cdots\int_{-t}^{t}dx_d\,e^{-2\pi i\sum h_jx_j}=\int_{-t}^{t}e^{-2\pi ih_1x_1}dx_1\cdots\int_{-t}^{t}e^{-2\pi ih_dx_d}dx_d=\prod_{j=1}^{d}\frac{\sin 2\pi h_jt}{\pi h_j}=$$

$$=\frac{1}{\pi^d}\frac{1}{N(h)}\prod_{j=1}^{d}\sin(2\pi h_jt).$$ So we want to bound $\sum_{h\in\Lambda^*\setminus\{0\}}\frac{1}{N(h)}\prod_{j=1}^{d}\sin(2\pi h_jt)e(h\cdot x)$ by $O\left((\log t)^{d-1}\right)$.

Ex. If Λ is admissible, so is Λ^*. Furthermore, $\exists$ func. s.t. $\eta(\delta)$ s.t. $N(\Lambda)\geq\delta\Rightarrow N(\Lambda^*)\geq\eta(\delta)$.

Convolutions and Smoothing

If $f,g:\mathbb{R}^d\to\mathbb{C}$ measurable, $(f*g)(x):=\int_{\mathbb{R}^d} f(x-y)g(y)\,dy$.

Properties ① If g is bounded, f is C^∞ and compactly supported then $f*g$ exists and is C^∞.

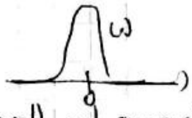
Idea of proof: differentiation under the integral sign.

② $f*g=g*f$

③ If f is "approximate identity" then $f*g\simeq g$, i.e. if $f=f_n\geq 0$, $\int_{\mathbb{R}^d} f_n\,dx=1\ (\forall n)$, $\mathrm{supp}(f_n)=\mathcal{U}_n$ a neighborhood of 0, and $\bigcap \mathcal{U}_n=\{0\}$, then $f_n*g\xrightarrow[n\to\infty]{} g$.

④ Let $\hat{f}(h)$ be $\int_{\mathbb{R}^d} f(x)\,e(-hx)\,dx$. Then $\widehat{(f*g)}=\hat{f}\cdot\hat{g}$

⑤ for $\tau>0$ define $w_\tau(x)=\frac{1}{\tau^d}w(\frac{x}{\tau})$ Then $\hat{w}_\tau(h)=\hat{w}(\tau h)$.



We'll use ①-⑤ in the following way. Let $w:\mathbb{R}^d\to[0,1]$, $\int_{\mathbb{R}^d} w(x)dx=1$, w is smooth and supported in a ball, and $w\equiv 1$ in a smaller ball around 0. Let $\tau>0$, and consider $\mathrm{supp}(w_\tau)$. It shrinks but always stays in a neighborhood of 0. Thus $\{w_\tau \mid \tau>0\}$ is an approximate identity.

$\hat{\chi}_\theta(h)\hat{w}_\tau(h)=\widehat{(\chi_\theta * w_\tau)}(h)$. Note: If $x\in\theta$, $\mathrm{dist}(x,\partial\theta)\geq\tau$, then $\chi_\theta * w(x)=1$. If $x\notin\theta$, $\mathrm{dist}(x,\theta)\geq\tau$, $(\chi_\theta * w)(x)=0$.

Conclusion $\chi_\theta * w$ is a smooth approximation of χ_θ which agrees with it for $\mathrm{dist}(x,\partial\theta)\geq\tau$.

#8 Recall Skriganov's Thm $\Lambda\subset\mathbb{R}^d$ a lattice, $\mathrm{covol}(\Lambda)=1$, admissible (i.e. $N_m(\Lambda)=\inf_{x\in\Lambda\setminus\{0\}}\{|x_1\cdots x_d|>0\}$

19.12.16 and θ is an axis-parallel box. Then $R(\theta-x,\Lambda)=\#(\theta-x)\cap\Lambda-\mathrm{Vol}(\theta)=c\log(3+\mathrm{Vol}(\theta)^{d-1})$, c depends only $N_m(\Lambda)$. (This what we're trying to prove)

Prop ("Smoothing"): For $\tau>0$, $\theta=[-t,t]^d$. $\theta_\tau^\pm=[-(t\pm\tau),(t\pm\tau)]^d$. $\theta_\tau^-\subseteq\theta_\tau\subseteq\theta_\tau^+$.

$R_\tau^\pm(\theta-x,\Lambda)=\sum_{h\in\Lambda^*\setminus\{0\}}\hat{\chi}_{\theta_\tau^\pm}(h)\hat{w}(\tau h)e(hx)\underset{\text{for all } x}{=}\left(\left[\#(\theta_\tau^\pm-\cdot)\cap\Lambda-\mathrm{Vol}(\theta_\tau^\pm)\right]*w_\tau\right)(x)$.

Then $R(\theta-x,\Lambda)\leq\mathrm{Vol}(\theta_\tau^+)-\mathrm{Vol}(\theta_\tau^-)+\sup_{x\in\mathbb{R}^d}\left[|R_\tau^+(\theta-x,\Lambda)|+|R_\tau^-(\theta-x,\Lambda)|\right]$.

Proof: $w_\tau*\chi_{\theta_\tau^\pm}$ is C^∞, compactly supported, non-negative. Also, $w_\tau*\chi_{\theta_\tau^\pm}(x)=\int_{\mathbb{R}^d} w_\tau(x-y)\chi_{\theta_\tau^\pm}(y)\,dy\in[0,1]$.

~~If $x\notin\theta$ nothing to prove. If $x\in\theta$, $\mathrm{supp}\{y\mapsto w_\tau(x-y)\}\subset B(x,\tau)\subset\theta_\tau^+$~~. Also $w_\tau*\chi_{\theta_\tau^-}\leq\chi_\theta\leq w_\tau*\chi_{\theta_\tau^+}$.

Let's prove the right ineq. If $x\notin\theta$ it's obvious. If $x\in\theta$, $\mathrm{supp}\{y\mapsto w_\tau(x-y)\}\subset B(x,\tau)\subseteq\theta_\tau^+$. So in the definition of $w_\tau*\chi_{\theta_\tau^+}(x)$, the integrand is always 1 when $w_\tau(x-y)\neq 0$. So $w_\tau*\chi_{\theta_\tau^+}(x)=1$. Define

$N_\tau^\pm(x)=\sum_{v\in\Lambda}(w_\tau*\chi_{\theta_\tau^\pm})(v+x)$. We have: $N_\tau^-(x)\leq\#(\theta-x)\cap\Lambda\leq N_\tau^+(x)$. By PSF, $N_\tau^\pm(x)=\mathrm{Vol}(\theta_\tau^\pm)+R_\tau^\pm(\theta-x,\Lambda)$.

$\#(\theta\cap(\Lambda+x))\leq N_\tau^+(x)\leq\mathrm{Vol}(\theta_\tau^+)+R_\tau^+(\theta-x,\Lambda)$, $\#(\theta\cap(\Lambda+x))\geq N_\tau^-(x)\geq\mathrm{Vol}(\theta_\tau^-)+R_\tau^-(\theta-x)$

$\mathrm{Vol}(\theta_\tau^+)-\mathrm{Vol}(\theta_\tau^-)-R_\tau^-(\theta-x,\Lambda)\leq\#\theta\cap(\Lambda+x)\leq\mathrm{Vol}(\theta_\tau^+)-\mathrm{Vol}(\theta_\tau^-)+R_\tau^+(\theta-x,\Lambda)$. ∎

We'll work with $\theta=[-t,t]^d$, $\tau=O(t^{d-1})$. Then $\mathrm{Vol}(\theta_\tau^+)-\mathrm{Vol}(\theta_\tau^-)=2(t+\tau)^d-2(t-\tau)^d=O(1)$.

Reduction to $\theta=[-t,t]^d$: For any $\theta'=T([-t,t]^d)$, T diagonal with positive entries.

Define $t>0$ by $t^d = N_m(T)$, i.e. $T=\begin{pmatrix} t & & \\ & \ddots & \\ & & t\end{pmatrix} a$, where $a\in A=\{\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_d\end{pmatrix} \mid \prod a_j = 1\}$.

$\#(\mathcal{O}'-x)\cap\Lambda = \#(a\begin{pmatrix} t & & \\ & \ddots & \\ & & t\end{pmatrix}[-1,1]^d - \underset{a a^{-1}}{x})\cap\Lambda = \#([-t,t]^d - a^{-1}x)\cap a^{-1}\Lambda$. Since $a^{-1}\Lambda$ is also admissible and $N_m(a^{-1}\Lambda)=N_m(\Lambda)$, $\mathrm{Covol}(a^{-1}\Lambda)=\mathrm{Covol}(\Lambda)=1$, and our estimates only depend on $N_m(\Lambda)$, it's enough to prove for $\mathcal{O}=[-t,t]^d$.

<u>Summary</u> Need to prove: If Λ admissible, $\mathrm{Covol}(\Lambda)=1$, $R_\varepsilon^{\pm}(\mathcal{O}-x,\Lambda) = c\log(t)^{d-1}$, where $\mathcal{O}=[-t,t]^d$, $t\geq 1$, c depends only on $N_m(\Lambda)$.

We're trying to bound $R_\varepsilon^{\pm}(\mathcal{O}-x,\Lambda) = \sum_{h\in\Lambda^*\setminus\{0\}} \hat{\chi}_{\mathcal{O}^{\pm}_\varepsilon}(h)\hat{\omega}(\varepsilon h)e(hx)$. Recall that ω is C^∞, supp in $B(0,1)$, const on $B(0,\frac{1}{2})$. Replace ω by a new function ω_1, C^∞, supp on $B(0,\frac{1}{\varepsilon^2})$, $\omega_1\equiv 1$ on $B(0,\frac{1}{2\varepsilon^2})$ (e.g. $\omega_1(x)=c_1\omega(\frac{x}{\varepsilon^2})$). i.e., we want to bound (*) $\sum_{h\in\Lambda^*\setminus\{0\}}\hat{\chi}_{\mathcal{O}}(h)\hat{\omega}_1(\varepsilon h)e(hx)$ by a bound of the form $c\log(t)^{d-1}$, $\mathcal{O}=[-t,t]^d$. One can divide (*) to two sums, depending only on a cut-off parameter ρ, $(*)=A_\rho+B_\rho$ (A_ρ, B_ρ func. of $x,\mathcal{O},\Lambda$, etc.):

$$A_\rho(x) = \sum_{h\in\Lambda^*\setminus\{0\}} \hat{\chi}_{\mathcal{O}}(x)\,\omega_2(\varepsilon h)\,\omega_1(\tfrac{h}{\rho})\,e(hx),\quad B_\rho(x) = \sum_{h\in\Lambda^*\setminus\{0\}}\hat{\chi}_{\mathcal{O}}(h)\,\omega_2(\varepsilon h)[1-\omega_1(\tfrac{h}{\rho})]e(hx),$$ where $\omega_2=\hat{\omega}_1$.

A_ρ is a finite sum, vanishes when $\|h\|\geq\frac{\rho}{\varepsilon^2}$. B_ρ is an infinite sum, if $\|h\|\leq\frac{\rho}{\varepsilon^2}$ the summand is 0.

We'll first bound B_ρ. Recall $\hat{\chi}_{\mathcal{O}}(h) = \frac{1}{N_m(h)}\prod_{j=1}^{d}\sin(2\pi h_j t)$ so $\hat{\chi}_{\mathcal{O}}(h) = \mathcal{O}(\frac{1}{N_m(h)}) = \mathcal{O}(1)$ since Λ^* is admissible. $1-\omega_1(\frac{\rho}{h})=\mathcal{O}(1)$. (Imp. const. depends on $N_m(\Lambda)$)

$\omega_2(y) = \hat{\omega}_1(y) = \int_{\mathbb{R}^d}\omega_1(x)e(-yx)dx = \underset{\text{for all } \alpha>0}{\mathcal{O}}(1+\|y\|)^{-\alpha})$, this is standard in Fourier Analysis (since ω_1 is C^∞ and comp. supp)

Let $\alpha>d^2$. Then $B_\rho = \mathcal{O}\Big(\sum_{\substack{h\in\Lambda^*\setminus\{0\}\\ \|h\|\geq\frac{\rho}{2\varepsilon^2}}}\omega_2(\varepsilon h)\Big) = \mathcal{O}\Big(\varepsilon^{-\alpha}\sum_{\substack{\|h\|\geq\frac{\rho}{2\varepsilon^2}\\ h\in\Lambda^*}}\|h\|^{-\alpha}\Big) = \mathcal{O}\Big(\varepsilon^{-\alpha}\sum_{p=\lfloor\log_2\frac{\rho}{\varepsilon}\rfloor} 2^{d(p+1)}2^{-p\alpha}\Big) =$

dividing the sum to contributions of $\|h\|\in[\frac{\rho}{\varepsilon^2}2^{p-1},\frac{\rho}{\varepsilon^2}2^p]$. Their number is bounded by $\mathcal{O}((\frac{\rho}{\varepsilon^2}2^p)^d)$ by trivial bounds in Gauss' circle problem and each is bounded by $\frac{\rho}{\varepsilon^2}2^p$.

$$= \mathcal{O}\Big(\varepsilon^{-\alpha}\sum_{p=\lfloor\,\rfloor}^{\infty}(2^{d-\alpha})^p\Big) = \mathcal{O}\Big(\varepsilon^{-\alpha}2^{(d-\alpha)\log_2\frac{\rho}{\varepsilon^2}}\Big) =$$

$$= \mathcal{O}(\varepsilon^{-\alpha}\rho^{d-\alpha}) = \mathcal{O}(t^{(d-1)\alpha+d(d-\alpha)}) = \mathcal{O}(t^{-\alpha+d^2}) = \mathcal{O}(1).$$

choose $\rho=t^d$, $\varepsilon=\mathcal{O}(t^{-(d-1)})$

Controlling A_ρ is the hardest part of the proof.

<u>Exercise:</u> If Λ is a lattice constructed as geometric embedding of the ring of integers $\mathcal{O}_K$ in a totally real field of deg d, then $\sum_{\substack{h\in\Lambda^*\\ 0<\|h\|\leq\rho}}\frac{1}{N_m(h)} = \mathcal{O}(\log(\rho)^d)$.

<u>Remarks</u> The conclusion of the [redacted] Ex. is true for any admissible Λ.

<u>Estimating A_ρ Using Ex.:</u> $\sum_{\substack{h\in\Lambda^*\setminus\{0\}\\ \|h\|\leq\frac{\rho^2}{\varepsilon}}}\frac{1}{N_m(h)}\prod\sin(2\pi h_j t)\,\omega_2(\varepsilon h)\omega_1(\frac{h}{\rho})e(hx) = \mathcal{O}\Big(\sum\frac{1}{|N_m(h)|}\Big) \overset{\text{ex.}}{=} \mathcal{O}(\log(\rho)^d) = \mathcal{O}(\log(t)^d)$

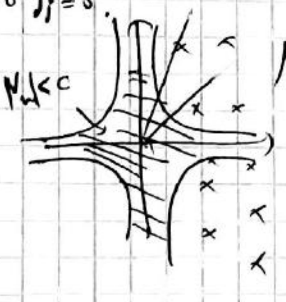
We get a bound of $\log(t)^d$ instead of $\log(t)^{2^d-1}$.

We would sketch the proof of controlling A_p. We want to think of A_p as $F(0)$

for $F(y)=\sum_{v\in\Lambda} f(y+v)$, where $f(y)=\frac{1}{N_m(y)}\prod_{j=1}^{d}\sin(2\pi y_j t)w_2(\tau y)w_1(\frac{y}{p})e(yx)$. This is

not smooth and generally bad, so we need to smooth it, and also $N_m(y)=0$ sometimes.
Note that it's compactly supported, since if $N(y)$ is big, $w_1(\frac{y}{p})=0$. To make sense of this,
we'll replace f with a new function which is well defined at $y_j=0$.

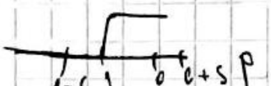
We'll decompose the sum into countably many sectors as drawn and sum up contributions in each sector.



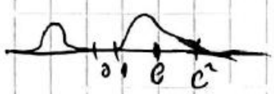
$N_m<c$ Λ^* admissible

Let $\beta:(1-\delta,e+\delta)\to\mathbb{R}$ be C^∞, $\beta(t)\in[0,1]$, $\delta>0$

$\beta|_{(1-\delta,1]}\equiv 0$, $\beta|_{[e,e+\delta)}\equiv 1$.



Let $\alpha(t)=\begin{cases}0 & |t|\le 1\\ \beta(t) & 1\le|t|\le e\\ 1-\beta(t) & e\le|t|\le e^2\\ 0 & |t|\ge e^2\end{cases}$



α is even.

$\sum_{q\in\mathbb{Z}}\alpha(e^{-q}t)=\begin{cases}1 & t\neq 0\\ 0 & t=0\end{cases}$. For $Q\in\mathbb{Z}^{d-1}$, set $e^{Q}=\begin{pmatrix}e^{a_1} & & & \\ & \ddots & & \\ & & e^{a_{d-1}} & \\ & & & e^{-\sum a_j}\end{pmatrix}\in A$.

Let
$M(x):=\prod_{j=1}^{d-1}\alpha(x_j)$ ~~$M_Q(x)=M(e^{Q}x)$~~, $M_Q(x):=M(e^{-Q}x)$.

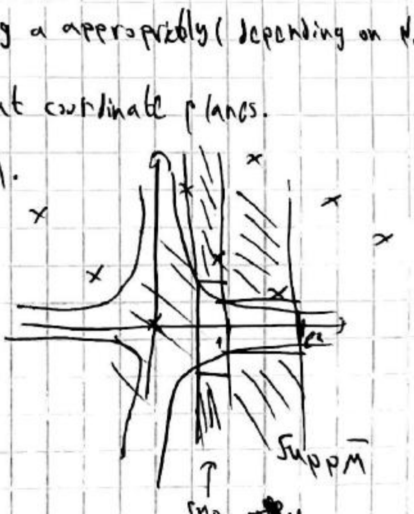
By induction, $\sum_{Q\in\mathbb{Z}^{d-1}}M_Q(x)=\begin{cases}1 & N_m(x)\neq 0\\ 0 & N_m(x)=0\end{cases}$. $\operatorname{Supp}M=\left(\prod_{j=1}^{d-1}\pm[1,e^2]\right)\times\mathbb{R}$.

Let η be an even function, C^∞, taking values in $[0,1]$ s.t. $\eta|_{[0,1]}\equiv 0$, $\eta|_{[C,\infty)}\equiv 1$.

$\bar{M}_Q(x):=M_Q(x)\eta(ax_d)$, $a>0$ a parameter. Choosing a appropriately (depending on $N_m(\Lambda)$),
$\forall h\in\Lambda^*\setminus\{0\}$, $\bar{M}_Q(h)=M_Q(h)$, and $\bar{M}_Q$ vanishes at coordinate planes.

Replace A_p by $A_p^{(Q)}=\sum_{h\in\Lambda^*\setminus\{0\}}\hat{k}_g(h)w_2(\tau h)w_1(\frac{h}{p})e(hx)\bar{M}_Q(h)$.

$\sum_{Q\in\mathbb{Z}^{d-1}}A_p^{(Q)}=A_p$.



Supp $\bar{M}$

supp M_Q

By PSF, we can express $A_p^{(Q)}(x)$ as

$\sum_{v\in V}\hat{F}(v)\cdot e(vy)$, when we evaluate at $y=0$.

$\hat{F}(v)=\int\frac{1}{N_m(Y)}\bar{M}_Q(Y)\prod_{j=1}^{d}\sin(2\pi y_j t)w_1(\frac{Y}{p})w_2(\tau Y)e(-vY)\,dY$

Main Lemma: This is $O\left(\prod_{j=1}^{d}(1+|v_j|)^{-\alpha}\right)$, imp. const. depends on $w_1,w_2,N_m(\Lambda)$ but not on t,Q,τ.

Additional Lemma (easier) $\sum_{v\in\Lambda,\,|v|\le\frac{p}{\tau}}\left(\prod(1+|v_j|)^{-\alpha}\right)=O(\log p)^{d-1}$.

26/12/16
#

Notation: For two functions $A(N), B(N)$, $A=\Omega(B) \iff \exists C>0$ s.t. $\forall N$ $|A(N)| \geq C|B(N)|$

Lower Bounds

Recall: $R_d = \{$axis parallel boxes in $\mathcal{U}^d\}$, $R_d^* = \{$axis parallel boxes in $\mathcal{U}^d$ with corner at $0\}$, $D((p_n)_{n=0}^{N-1}; R_d) = \sup_{B \in R_d} N\,\mathrm{Vol}(B) - \#\{n<N \mid p_n \in B\}$

Thm 1 (Roth '54) $\forall N$, $\forall (p_n)_{n=0}^{N-1} \subset \mathcal{U}^2$ $D((p_n)_{n=0}^{N-1}; R_2) = \Omega(\sqrt{\log N})$.

Thm 1' (Roth '54) $\forall d$ $\forall N$ $\forall (p_n)_{n=0}^{N-1} \subset \mathcal{U}^d$, $D((p_n)_{n=0}^{N-1}; R_d) = \Omega(\log(N)^{\frac{d-1}{2}})$.

Recall that there are sets $(p_n)_{n=0}^{N-1} \subset \mathcal{U}^d$ s.t. $D((p_n)_{n=0}^{N-1}; R_d) = O(\log(N)^{d-1})$, so the bound is not tight. However:

Thm 2 (Schmidt 72) $\forall N$ $\forall (p_n)_{n=0}^{N-1} \subset \mathcal{U}^2$, $D((p_n)_{n=0}^{N-1}); R_2) = \Omega(\log N)$.

Cor. In dimension 2, Schmidt's theorem is optimal up to constants.

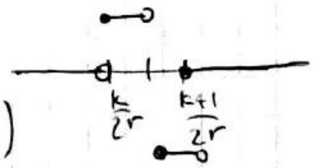
Remark: The correct rate of growth is not known. Beck and Chen call it "the great open problem of discrepancy". Best known result to date: (Bilyk-Lacey-Vagharshakyan, following Beck, '08): $\forall d$ $\exists \eta(d)$ $\forall N$ $\forall (p_n)_{n=0}^{N-1} \subset \mathcal{U}^d$, $D((p_n)_{n=0}^{N-1}; R_d) = \Omega(\log(N)^{\frac{d-1}{2}+\eta(d)})$. Argument shows (maybe? Barak is not sure) $\eta(d) = \Omega(\frac{1}{d^2})$.

Conjectures: Several authors conjectured that the correct rate is $(\log N)^{d-1}$. Recently Bilyk listed 2 conjectures: * correct rate is $\log(N)^{\frac{d}{2}}$

* correct rate is $\log(N)^{\frac{d-1}{2}+\frac{d-1}{2}}$ (attributed to Skriganoff).

Preparations for Proofs of 1,2

For $r \in \mathbb{N} \cup \{0\}$, $k \in \mathbb{Z}$, define
$$\psi_{k,r}(t) = \begin{cases} 0 & t \notin [\frac{k}{2^r}, \frac{k+1}{2^r}) \\ +1 & t \in [\frac{k}{2^r}, \frac{k}{2^r} + \frac{1}{2^{r+1}}) \\ -1 & t \in [\frac{k}{2^r} + \frac{1}{2^{r+1}}, \frac{k+1}{2^r}) \end{cases}$$



Remark up to scaling and up to values at discontinuities, these func. form the Haar orthogonal system.

In dim=2, for $r=(r_1,r_2)$, $k=(k_1,k_2)$, $k_i \in \{0,\dots,2^{r_i}-1\}$, we'll use $\psi_{k,r}(x,y) = \psi_{k_1,r_1}(x)\psi_{k_2,r_2}(y)$

($\psi_{k,r} = \psi_{k_1,r_1} \otimes \psi_{k_2,r_2}$).

Example $d=2$
$r_1=3$, $r_2=1$
$k_1=3$, $k_2=0$

Note that $\mathrm{Supp}\,\psi_{r,k} = [\frac{k_1}{2^{r_1}}, \frac{k_1+1}{2^{r_1}}) \times [\frac{k_2}{2^{r_2}}, \frac{k_2+1}{2^{r_2}})$.

~~Prop If $(r,k)\neq(s,l)$ then $\langle \psi_{r,k}, \psi_{s,l} \rangle := \int \psi_{r,k}(x)$...~~

Prop If $(r,k) \neq (s,l)$ then $\langle \psi_{r,k}, \psi_{s,l} \rangle := \int_{\mathcal{U}^2} \psi_{r,k}(x)\psi_{s,l}(x)\,dx = 0$.

Pf Case 1. $r=s$, $k \neq l$. Then their supports are disjoint (except maybe the boundaries) and it is trivial.

If $r \neq s$, w.l.o.g. $r_1 > s_1$. For fixed y, $x \mapsto \psi_{s,\ell}(x,y)$ is constant on its intersection with $\mathrm{supp}(\psi_{r,k})$, and so $\psi_{r,k}\psi_{s,\ell}$ assumes ± 1 on subintervals of equal length. Hence for any y, $\int_0^1 \psi_{r,k}(x,y)\psi_{s,\ell}(x)\,dx = 0$. $\Rightarrow$ ∎

The same proof gives "generalized orthogonality": If $f_1, \ldots, f_j$ are distinct func. of the form $\psi_{r,k}$, then $\int f_1 \cdots f_j \, dx = 0$.

Remark By a similar idea, if φ_n, $n \in \mathbb{Z}$ are orthogonal then $\varphi_{n,m} = \varphi_n \otimes \varphi_m$ are orthogonal.

Remark: After normalizing, $\psi_{r,k}$ are an orthonormal basis for $L^2(\mathcal{U}^2)$.

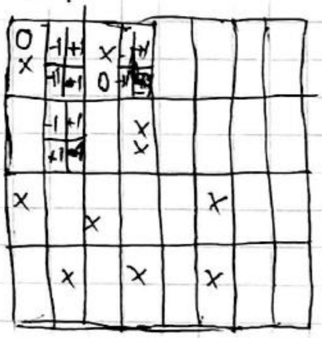
Pf of Thm 1: Given $(p_n)_{n=0}^{N-1}$, for $x = (x_1, x_2) \in \mathcal{U}^d$, define $D(x) = Nx_1x_2 - \#\{n < N \mid p_n \in [0,x_1) \times [0,x_2)\}$.
Note that $\|D\|_\infty = D((p_n)_{n=0}^{N-1}; R_2^*)$. It suffices to bound $\|D\|_\infty$ from below.

~~We'll define F~~ We'll define $F : \mathcal{U}^2 \to \mathbb{R}$ (depending on (p_n)) s.t. (i) $|\langle D, F\rangle| = \Omega(\log N)$
(ii) $= \|F\|_2 = O(\sqrt{\log N})$,

and the implicit constants don't depend on N, (p_n). Thus we'll get $\|D\|_\infty \geq \|D\|_2 \geq \frac{|\langle D, F\rangle|}{\|F\|_2} = \frac{\Omega(\log N)}{O(\sqrt{\log N})} = \Omega(\sqrt{\log N})$.

Chose m so that $2N \leq 2^m < 4N$, so $m \asymp \log(N)$, where $A \asymp B \Leftrightarrow (A = O(B) \wedge A = \Omega(B))$.

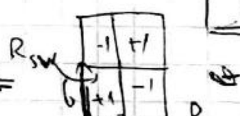
Let $F = \sum_{j=0}^{m} f_j$, where $f_j = \sum_{\{k \mid \mathrm{supp}\,\psi_{r_j,k} \text{ contains no pts of } (p_n)\}} \psi_{r_j,k}$, $r_j = (j, m-j)$



Picture $d=2$, $N=10 \Rightarrow m=5$, $2^m = 32$. $j=3$, $m-j=2$.

Now, $\langle D, f_j\rangle = \int_{\mathcal{U}^2} D(x) f_j(x)\,dx = \sum_{R \text{ contains no pts of } (p_n)} \int_R D f_j \,dx$.

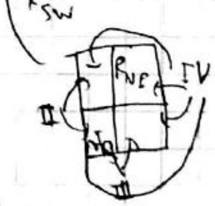
$\int_R D f_j \,dx = \int_{R_{SW}} [D(x) + D(x+a+b) - D(x+a) - D(x+b)]\,dx =$



$= \int_{R_{SW}} [Nx_1x_2 + N(x_1 + 2^{-(j+1)})(x_2 + 2^{-(m-j+1)}) - N(x_1 + 2^{-(j+1)})x_2 - Nx_1(x_2 + 2^{-(m-j+1)})]$

$- \int_{R_{SW}} \#\{n \mid p_n \in [0,x_1) \times [0,x_2)\} + \#\{\ldots\} - \#\{\ldots\} - \#\{\ldots\} = \int_{R_{SW}} [N\mathrm{Vol}(R_{NE}) - \#\{n \mid p_n \in R_{NE}\}]\,dx$

$= N\mathrm{Vol}(R_{SW})^2 - 0 = N(2^{-(j+1)}2^{-(m-j+1)})^2 = \frac{N}{16}2^{-2m}$ (*)



$\mathrm{I} + \mathrm{IV} - \mathrm{II} - \mathrm{III} = R_{NE}$

Therefore $\int_{R_{SW}} Df_j \geq \frac{N}{16}2^{-2m}$. Thus $\int_{\mathcal{U}^2} Df_j \geq \frac{N \cdot N 2^{-2m}}{16} = \Omega(1)$ ← we sum only on boxes with no points in them, and there are at least N of these.

Thus, $\langle D, F\rangle = \sum_{j=0}^{m} \langle D, f_j\rangle = m\Omega(1) = \Omega(\log(N))$.
(all summands are ≥ 0)

Now, $\|F\|_2^2 = \langle F, F\rangle = \langle \sum_{j=0}^m f_j, \sum_{k=0}^m f_k\rangle = \sum_{j=0}^m \|f_j\|^2 + \sum_{j \neq k} \langle f_j, f_k\rangle = \sum_{j=0}^m \|f_j\|^2 = mO(1) = O(m) \Rightarrow \|F\|_2 = O(\sqrt{m}) = O(\sqrt{\log N})$. ∎

Note: in the proof of (*) we only needed that $F_j|_R = \boxed{\begin{smallmatrix}-1 & +1\\ +1 & -1\end{smallmatrix}}$ and no points of (p_n) in R.

An idea that was used: we considered $\frac{\langle D, F\rangle}{\|F\|_2}$ = length of the projection of D onto $\mathrm{span}(F)$.

F was a linear combination of $\psi_{r,k}$, where r was chosen so that "scale it picks up" is determined by cells of volume $\simeq \frac{1}{2N}$. On this scale, lots of pieces with no points. Having no points is used to ensure that D has large variation on each cell.

Proof of Thm 2 (Halász '81). Instead of Cauchy-Schwartz, we'll use

$$|\langle G, D\rangle| \le \|D\|_\infty \int_{\mathcal{U}^2} |G(x)|\,dx.$$ We'll choose D so that $\|D\|_\infty \ge \frac{|\langle G, D\rangle|}{\int_{\mathcal{U}^2} |G(x)|\,dx}$ is large.

We'll show $\langle G, D\rangle = \Omega(\log N)$, $\int |G| = O(1)$.

G will depend on a parameter $c \in (0, \frac{1}{2})$, and we'll use $G = (1 + cf_0)\ldots(1 + cf_m) - 1$, where m, f_j as in the previous proof. We can write $G = \sum_{k=1}^{m} G_k$, $G_k = c^k \sum_{0 \le j_1 < \ldots < j_k \le m} f_{j_1} \cdots f_{j_k}$. From multi-orthogonality, we get that $\int G_k = 0$. ($f_{j_1}, \ldots, f_{j_k}$ have different r's.)

$$\int_{\mathcal{U}^2} |G|\,dx \le \int_{\mathcal{U}^2} 1 + \int |\underbrace{(1 + cf_0)\ldots(1 + cf_m)}_{\ge 0}|\,dx = 1 + \int (1 + cf_0)\ldots(1 + cf_m)\,dx = 1 + 1 + \int \sum G_k = 2 = O(1).$$

So it remains to show that $\langle G, D\rangle = \Omega(\log(N))$.

$|\langle D, G\rangle| \ge |\langle D, G_1\rangle| - \sum_{k \ge 2} |\langle D, G_k\rangle|$. We'll bound $|\langle D, G_1\rangle|$ from below and $|\langle D, G_k\rangle|, k \ge 2$ from above.

$G_1 = cF$, where F is as in the previous proof, so $\langle D, G_1\rangle = c\,\Omega(\log(N))$. Now let $k \ge 2$, $0 \le j_1 < \ldots < j_k \le m$.

Let R be a dyadic box with sidelengths 2^{-j_k} (x-direction), $2^{-(m-j_1)}$ (y-direction). (These are the smallest sidelengths in the definition of $f_{j_1}, \ldots, f_{j_k}$). So on R, $f_{j_1} \cdots f_{j_k}$ is either 0 or $\pm \boxed{\begin{smallmatrix}-1 & +1\\ +1 & -1\end{smallmatrix}}$, and if it's of the second form, then R contains no points of (p_n). Thus $\int_R D f_{j_1} \ldots f_{j_k}\,dx = O(N\,\mathrm{Vol}(R)^2)$, by (*) in the previous proof (and the remark on top of this page).

$$= O\left(N\,\mathrm{Vol}(R) \cdot \frac{1}{2^{j_k}} \frac{1}{2^{m-j_1}}\right).$$

Write $q = j_k - j_1$, and sum over all R: $\int_{\mathcal{U}^2} D f_{j_1} \ldots f_{j_k} \underset{\sum \mathrm{Vol}(R) \le 1}{=} O\left(N \frac{1}{2^{m+q}}\right) = O(2^{-q})$.

$$\sum_{k=2}^{m} \langle D, G_k\rangle \le \sum_{k=2}^{m} c^k \sum_{\substack{0 \le j_1 < \ldots < j_k \le m\\ q = j_k - j_1}} O(2^{-q}) = \sum_{k=2}^{m} c^k \sum_{q=k-1}^{m} \underbrace{(m-q+1)}_{\text{choosing } j_1} \underbrace{\binom{q-1}{k-2}}_{j_2, \ldots, j_{k-1}} O(2^{-q}) \le O(1)\,m \sum_{q=1}^{m} 2^{-q} \sum_{k=2}^{q+1} \binom{q-1}{k-2} c^k$$

$$\underset{j = k-2}{\le} O(1)\,m \sum_{q=1}^{m} 2^{-q} c^2 \sum_{j=0}^{q-1} \binom{q-1}{j} c^j \le O(1)\,m \sum_{q=1}^{m} 2^{-q} c^2 (1+c)^{q-1} \le O(1)\,m c^2 \cdot 2^{-1} \sum_{q=1}^{m} \left(\frac{1+c}{2}\right)^{q-1} \le$$

$$\underset{c \le \frac{1}{2}}{\le} c^2\,O(\log N).$$

$$\Longrightarrow |\langle D, G\rangle| \ge |\langle D, G_1\rangle| - \sum_{k=2}^{m} \langle D, G_k\rangle = c\,\Omega(\log N) - c^2\,O(\log N) \underset{c \text{ small}}{=} \Omega(\log N)$$

$\sum_{q=1}^{m} \left(\frac{1+c}{2}\right)^{q-1} \le \sum_{r=0}^{\infty} \left(\frac{1+c}{2}\right)^r \le 4 = O(1)$

$\Longrightarrow$ Q.E.D.

Remark: The proof of Roth's theorem was via a lower bound for $\|D\|_2$. In discrepancy, one can study upper and lower bounds for $\|D\|_p$, $1 \le p \le \infty$.

~~Upper bou~~ Lower bounds: Schmidt for $1<p<\infty$ $\|D\|_p \geq \Omega(\log(N)^{\frac{d-1}{2}})$ ← imp. const. dep. on p

Upper bounds: Skriganoff (same paper) showed how to construct points from admissible lattices leading to $O(\log(N)^{\frac{d-1}{2}})$ for $1<p<\infty$. The correct rate of growth for $\|D\|_1, \|D\|_\infty$ is unknown.

$\frac{8}{9}.1.16$ #

Recall that for $\mathcal{R} \subset 2^{\mathcal{U}^d}$, $D(N;\mathcal{R}) = \inf_{(x_n)_{0\le n \le N}} D((x_n)_{n=0}^{N-1}; \mathcal{R})$.

Examples for which $D(N)$ behaves like N^δ for some $\delta>0$ (i.e. not a power of $\log N$):

① $\mathcal{R} = \{$intersections of $\mathcal{U}^d$ with half-~~sp~~space$\}$. Then $D(N;\mathcal{R}) = N^{\frac{1}{2}-\frac{1}{2d}}$ (up to a constant)

② $\mathcal{R} = \{$balls in $\mathcal{U}^d\}$, $D(N;\mathcal{R}) = O(N^{\frac{1}{2}-\frac{1}{2d}}(\log N)^{\frac{1}{2}}) = \Omega(N^{\frac{1}{2}-\frac{1}{2d}})$

Typically lower bounds use Fourier analysis, but in one case there's a geometric ~~argum~~ argument.

Let $\mathcal{A} = \{$convex subsets of $\mathcal{U}^2\}$.

Thm (Schmidt '75 "chocolate cake argument"): $D(N;\mathcal{A}) = \Omega(N^{\frac{1}{3}})$, i.e. $\exists c>0$ $\forall N$ $\forall x_0, \dots, x_{N-1} \in \mathcal{U}^2$, $D((x_n)_{n=0}^{N-1}; \mathcal{A}) \geq cN^{\frac{1}{3}}$.

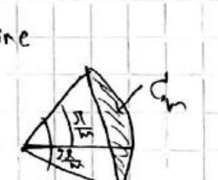
Remarks: ① There's an upper bound which almost matches: (Beck '87) $D(N;\mathcal{A}) = O(N^{\frac{1}{3}}(\log N)^4)$

② A similar argument to Schmidt's, for $d>2$, gives $D(N;\mathcal{A}_d) = \Omega(N^{1-\frac{2}{d+1}})$.

Pf: Idea:  throwing out any of these small domains still gives a convex set. We choose parameters so that each of the c_j has $\frac{1}{2}$pt. on average.

Let's rescale to assume $\mathcal{U}^2 = [-1,1]^2$. Let $m \in \mathbb{N}$. For $j = 1, \dots, m$, define

$C_j' := \{(x,y) \in \mathcal{U}^2 \mid x^2+y^2 \le 1, \langle \binom{x}{y}, \binom{\cos\frac{2\pi j}{m}}{\sin\frac{2\pi j}{m}} \rangle \geq \cos(\frac{\pi}{m})\}$.



So $\mathrm{area}(C_j') = \frac{\pi}{m} - \cos(\frac{\pi}{m})\sin(\frac{\pi}{m}) = \frac{\pi}{m} - (1-\frac{1}{2}(\frac{\pi}{m})^2 + \dots)(\frac{\pi}{m} - \frac{1}{6}(\frac{\pi}{m})^3 + \dots) =$

$= \frac{2}{3}(\frac{\pi}{m})^3 + O(\frac{1}{m^4})$.

Let $(x_n)_{n=0}^{N-1}$ be N pts. in $\mathcal{U}^2$. Choose m so that on average, each C_j' contains $\frac{1}{2}$ a point from $(x_n)_{n=0}^{N-1}$, i.e. $\frac{1}{4}\frac{2}{3}(\frac{\pi}{m})^3 \cdot N = \frac{1}{2}$, so $m = \lfloor (\frac{1}{3}\pi^3 N)^{\frac{1}{3}} \rfloor$. With this m, there are c_1, c_2, $0<c_1<\frac{1}{2}<c_2<1$, s.t. for each j, $\mathrm{Vol}(C_j') = [\frac{c_1}{N}, \frac{c_2}{N}]$ (c_1, c_2 don't depend on N for large N). Now, given $(x_n)_{n=0}^{N-1}$ set $J_1 = \{j \mid C_j$ contains none of the $(x_n)\}$, $J_2 = \{j \mid C_j$ contains at least one $x_n\}$. Let $B = \{x^2+y^2 \le 1\}$, and $B_1 = B \setminus \bigcup_{j \in J_1} C_j'$, $B_2 = B \setminus \bigcup_{j\in J_2} C_j'$. B, B_1, B_2 are convex. Let $c>0$, $c<\frac{c_1}{4}$, $c<\frac{1-c_2}{4}$. We'll show that at least one of B, B_1, B_2 has discrepancy $\geq cm = \Omega(N^{\frac{1}{3}})$. Write $D(B) = N\mathrm{Vol}(B) - \#\{n<N, x_n \in B\}$, and $D(B_1), D(B_2)$ similarly. So $D(B_1) = N\mathrm{Vol}(B_1) - \#\{n<N \mid x_n \in B_1\} = N\mathrm{Vol}(B) - N|J_1|\mathrm{Vol}(C_j) - \#\{n<N \mid x_n \in B\} =$

(1) $= D(B) - |J_1|c_1 = D(B)$

(2) $D(B_2) = \dots \geq D(B) - N|J_2|\mathrm{Vol}(C_j') + |J_2|$

If $D(B)\ge cm$ we're done, so assume $D(B)<cm$. If $|J_1|>\frac{m}{2}$, then apply (1) and get:

$$|D(B_1)|\ge |J_1|c_1 - D(B)\ge \frac{m}{2}c_1 - cm \ge m\left(\tfrac{1}{2}c_1-\tfrac{1}{4}c_1\right)\ge mc$$

If $|J_2|\ge \frac{m}{2}$,

$$|D(B_2)| = |J_2|(1-N\mathrm{Vol}(d_j)) - cm \ge |J_2|(1-c_2)-cm\ge cm. \ \blacksquare$$

Let $B_d = \{\text{closed balls in } \mathbb{R}^d\}$.

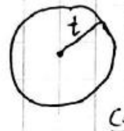
Thm A (From Matousek's book) $D(N;B_2)=O(N^{\frac14}\sqrt{\log N})$.

Remarks (1) In higher dimensions $d>2$, similar arguments give $D(N;B_d)=O(N^{\frac12-\frac1{2d}}\sqrt{\log N})$.

(2) Up to a log term, there's a matching lower bound. The true asymptotics are not known.

(3) The upper bound will involve a probabilistic construction, we won't have an explicit placement of N pts with low discrepancy, rather we'll define a prob. space and show that the existence of a placement of N pts has positive probability.

(4) Exponent $N^{\frac14}$ is consistent with Gauss circle problem:



$B(0,t)\cap\mathbb{Z}^2=\pi t^2+E(t)$

$E(t)\ge\Omega(t^{\frac12})$

Conj: $E(t)=O_\varepsilon(t^{\frac12+\varepsilon})\ \forall\varepsilon>0$.

We'll prove some lemmas in preparation for Thm A.

Lemma 1 (Chernoff bound) Suppose $X_1,\dots,X_m$ are independent random variables, where $X_i=\begin{cases}-p_i & \text{with prob. } 1-p_i\\ 1-p_i & \text{" } p_i\end{cases}$, for $p_i\in(0,1)$. (Note that $\mathbb{E}(X_i)=0$.) Then $\forall\Delta>0$, $\Pr\left(\sum_{i=1}^m X_i\ge\Delta\right)\le 2e^{-\frac{2\Delta^2}{m}}$

Sublemma 1: $\forall\alpha,\beta\in\mathbb{R}$, $|\alpha|\le 1$: $\cosh(\beta)+\alpha\sinh(\beta)\le e^{\frac{\beta^2}{2}+\alpha\beta}$

Pf: Ex. (Study RHS − LHS ...)

Sublemma 2 $\forall\theta\in[0,1], \forall\lambda\in\mathbb{R}$, $\theta e^{\lambda(1-\theta)}+(1-\theta)e^{-\lambda\theta}\le e^{\frac{\lambda^2}{8}}$.

Pf: Write $\theta=\frac{1+\alpha}{2}$, $\lambda=2\beta$ and use sublemma 1 (Ex.).

Pf of Lemma 1 For any $\lambda>0$, by sublemma 2, $\mathbb{E}(e^{\lambda X_i})=(1-p_i)e^{-\lambda p_i}+p_ie^{\lambda(1-p_i)}\le e^{-\frac{\lambda^2}{8}}$

So for $X=\sum X_i$, $\mathbb{E}(e^{\lambda X})=\mathbb{E}(\prod e^{\lambda X_i})=\prod\mathbb{E}(e^{\lambda X_i})\le e^{\frac{\lambda^2 m}{8}}$.

Thus $\Pr(X>\Delta)=\Pr(e^{\lambda X}>e^{\lambda\Delta})\underset{\text{Markov}}{\le}\frac{\mathbb{E}(e^{\lambda X})}{e^{\lambda\Delta}}\le e^{\frac{\lambda^2 m}{8}-\lambda\Delta}$. Take $\lambda=\frac{4\Delta}{m}$. Then:

$\Pr(X>\Delta)\le e^{-\frac{2\Delta^2}{m}}$. $\blacksquare$

Lemma 2 Let $\mathcal{A}\subset\mathbb{R}^d$ be a collection of sets. Let $b\ge 2$, $b\in\mathbb{N}$. Suppose that for $k=b^m$, $D(k;\mathcal{A})\le f(k)$ where f is a non-decreasing power. Then $\forall N$ $D(N;\mathcal{A})=O(f(1)+f(b)+\dots+f(b^r))$ where $b^r\le N\le b^{r+1}$.
implicit const depends on b, not on N

Pf Write $N=\sum_{i=0}^r a_ib^i$, $a_i\in\{0,\dots,b-1\}$. For $A\in\mathcal{A}$, $(x_n)_{n=0}^{N-1}\subset\mathbb{R}^d$:

$$|D((x_n)_{n=0}^{N-1};A)| = |N\mathrm{Vol}(A)-\#\{n<N \mid x_n\in A\}| = \left|\left(\sum_{i=0}^r a_ib^i\right)\mathrm{Vol}(A)-\#\{n<N \mid x_n\in A\}\right|\le\sum_{i=0}^r\left|a_ib^i\mathrm{Vol}(A)-\#\left\{\sum_{j=0}^{i-1}a_jb^j\le n<\sum_{j=0}^{i}a_jb^j,\ x_n\in A\right\}\right|$$

$\leq \sum_{i=0}^{r} \sum_{k=0}^{a_i-1} \left| b^i \mathrm{Vol}(A) - \#\{ \Sigma_{-} + kb^i \leq n < \Sigma_{-} + (k+1)b, x_n \in A\} \right| \leq \sum_{i=0}^{r} \sum_{k=0}^{a_i-1} f(b^i) \leq \sum_{i=0}^{r} (b-1) f(b^i) = O\left(\sum_{i=0}^{r} f(b^i)\right)$. ∎

<u>Cor</u> Suppose $c_1 > 0$, $c_2 \geq 0$, $A \in 2^{U^d}$. Then in order to prove $|D(N;A)| \leq O(N^{c_1} (\log N)^{c_2})$, it suffices to prove $|D(b^r;A)| \leq O(b^{rc_1} \log (b^r)^{c_2})$ for $b \in \mathbb{N}, b \geq 2$, and all r.

<u>Pf:</u> Define $f(N) = c_3 N^{c_1} (\log N)^{c_2}$, $c_1 > 0$, $c_3 > 0$, $c_2 \geq 0$. and use Lemma 2:

If $b^r \leq N < b^{r+1}$, $|D(N;A)| = O(f(1) + f(b) + \dots + f(b^r)) = O\left(\sum_{i=0}^{r} b^{ic_1} (\log i)^{c_2}\right) = O\left(\log b^{rc_2}\, b^{rc_1} \sum_{i=0}^{r} b^{(i-r)c_1}\right) =$
$= O((\log N)^{c_2} N^{c_1})$. ∎

geo. sum, $\left(\sum_{i=0}^{r} \dots \leq \sum_{i=0}^{\infty} < \infty\right)$

<u>Lemma 3</u> Let $Q \subset \mathbb{R}^2$ be a set of m points. We say $B_1, B_2 \in \mathcal{B}_2$ are <u>Q-equivalent</u> if $Q \cap B_1 = Q \cap B_2$. Then there are $O(m^3)$ equivalence classes in $\mathcal{B}$.

We'll discuss more general results that imply Lemma 3. The proof of lemma 3 is elementary and geometric.

<u>Pf</u> Say that $Q \subset \mathbb{R}^2$ is <u>generic</u> if no 4 points of Q lie on a circle, and no 3 pts of Q lie on a line.

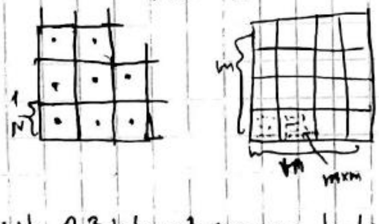
<u>Claim</u> We can assume wlog that Q is generic.

<u>Pf:</u> Suppose Q is general, and there are k equivalence classes of balls in $\mathcal{B}_2$. Chose $B_1, \dots, B_k \in \mathcal{B}_2$ to be representatives ($B_i \cap Q \neq B_j \cap Q$). By increasing the radii, we can assume $B_i \cap Q \subset \mathrm{int}(B_i)$. So for each $q \in Q$, $\bigcap_{q \in B_i} B_i \setminus \bigcup_{q \notin B_j} B_j$ contains a nbd of q. Moving each q in such a nbd does not change the set $\{i \mid q \in B_i\}$. So we can perturb Q to a generic Q', and $B_i \cap Q' \neq B_j \cap Q'$ for $i \neq j$.
So, Q' is generic and has $\geq k$ equivalence classes, so it suffices to bound for Q'.

We'll continue the proof of Lemma 3 next time.

<u>Pf of Thm A</u> By Lemma 2, we can assume $N = 4^k = (2^k)^2$, so we can write $N = m^2$.

Let $Q = \left[\left(\frac{1}{2N}, \frac{1}{2N}\right) + \frac{1}{N}\mathbb{Z}^2\right] \cap U^2$. Then $\#Q = N^2$. ~~Let P be a random ind. uniform choice of a pt in each square.~~ By the argument giving a trivial bound in the Gauss circle problem, for $B \in \mathcal{B}_2$, $|D(Q;B)| = |N^2 \mathrm{Vol}(B) - \#Q \cap B| = O(N)$. (**)



This is not a good bound, we'll now improve it. Since $N = m^2$, divide U^2 into m^2 squares, denote these squares by $\mathcal{G}$. For each $G \in \mathcal{G}$, $Q_G = Q \cap G$ contains N points. Choose one uniformly at random, ind. for each G, and denote it by q_G. The collection $P = \{q_G \mid G \in \mathcal{G}\}$ has N pts.
Let $C > 0$ be a paramater, we'll see how to chose it later. Let $\Delta = C N^{\frac{1}{4}} \sqrt{\log N}$. We're going to prove that with positive probability, $\forall B \in \mathcal{B}_2$ (*) $\left|\frac{1}{N} \# Q \cap B - \# P \cap B\right| \leq \Delta$.
Now suppose we have (*) for any $B \in \mathcal{B}_2$. Then $|N \mathrm{Vol}(B) - \# P \cap B| \leq \left|N \mathrm{Vol}(B) - \frac{\# Q \cap B}{N}\right| + \left|\frac{\# Q \cap B}{N} - \# P \cap B\right|$
$\leq O(1) + \Delta = O(\Delta)$. So it's enough to prove (*) $\forall B \in \mathcal{B}_2$. Let's analyze the prob. of (*) for a fixed $B \in \mathcal{B}_2$.
(**)
We'll analyze the contribution of each $G \in \mathcal{G}$ to the LHS of (*). Fix $G \in \mathcal{G}$, and let $k_G = \# B \cap Q_G$.

So $\frac{k_G}{N}$ is the contribution of G to $(*)$. q_G contributes either 1 or 0 to $\# P \cap B$, whether $q_G \in B$ or $q_G \notin B$.

Denote $X_G \begin{cases} -\frac{k_G}{N} & q_G \notin B \\ 1 - \frac{k_G}{N} & q_G \in B \end{cases}$. X_G is the RV measuring the contribution of G to $(*)$. Denote $X = \sum_{G \in \mathcal{G}} X_G$.

It's the LHS of $(*)$. Note that $\Pr(q_G \in B) = \frac{k_G}{N}$. If $G \subset B$ or $G \cap B = \emptyset$,

Then $X_G = 0$. So we're only interested in G s.t. $G \cap \partial B \neq \emptyset$. There are $O(m)$ of these, again by arguments

similar to the trivial bounds in Gauss Circle Problem. So by Lemma 1, $\Pr(X > \Delta) \leq 2e^{-\frac{2\Delta^2}{c_1 m}} = 2N^{-c_2}$,

where c_2 depends on c_1 and on C', and $c_2 \to \infty$ when $C \to \infty$. $\frac{\Delta^2}{c_1 m} = \frac{C^2 \sqrt{N} \log N}{c_1 \sqrt{N}}$, $e^{-\frac{2\Delta^2}{c_1 m}} = N^{-c_2}$.

So for fixed B, $\Pr[X > \Delta] \longrightarrow 0$ as $C \to \infty$.

If $B_1 \sim B_2$, $(*)$ doesn't change if $B = B_1$ or $B = B_2$. Let $\mathcal{F}$ be a collection of representatives.

It's enough to check $(*)$ for $B \in \mathcal{F}$, $\# \mathcal{F} = O(|\Omega|^3) = O(N^6)$.

So $\Pr[(*)$ fails for some $B \in \mathcal{B}_2] = \Pr[(*)$ fails for $B \in \mathcal{F}] \leq \sum_{B \in \mathcal{F}} \Pr[(*)$ fails for $B] \leq O(N^6) 2N^{-c_2}$.

For C sufficiently large, $c_2 > 6$ and so the prob. will be < 1, so with positive prob., $(*)$ holds for all $B \in \mathcal{B}_2$. $\boxtimes$