Lecture 5: 18/3/2018

The Arc-Complex

Def:

A simplicial complex X on vertice set Σ, is a collection of finite subsets of Σ s.t if $E \subseteq F \in X \Rightarrow E \in X$

Whenever $E \subseteq F \in X$, E is called a face of F.

The geometric realization of X consists of the union of $\{\Delta_E \mid E \in X\}$, Δ_E is a (closed) Euclidean simplex of dimension $|E|-1$.

Whenever $E \subseteq F$ we identify Δ_E with the corresponding face of Δ_F.

The star of $E \in X$ is the subcomplex consisting of

$$\{\Delta_F \mid E \subseteq F\}$$

The link of $E \in X$ is the subcomplex consisting of

$$\{\Delta_A \mid A \cap E = \phi,\ A \cup E \in X\} = \{\Delta_A \mid A = F \setminus E,\ F \in \mathrm{Star}(E)\}$$

Thm (Harer 85'):

$A(S,V)$ is either: (i) empty

or: (ii) homeomorphic to a sphere if S=disc, $V \subseteq \partial S$, $|V| \geq 4$

or: (iii) Contractible otherwise

Cor:

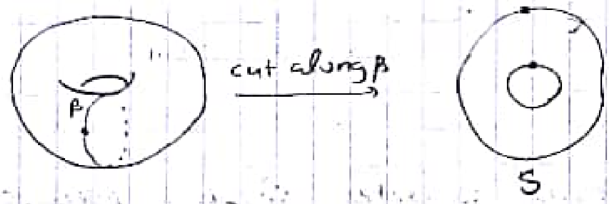
Every two triangulations of (S,V) are connected by a finite sequence of moves

Pf of corollary:

Lemma:

If E is a simplex in $A(S,V)$ of codim ≥ 2 then link (E) is connected.

For example :



then link(β) = $A(S,V)$ $\cong$ 

<u>pf of cor.</u> :

By the thm. $\exists$ path in $A(S,V)$ between any two top-dim simplices. We need to show there is a path passing only through top-dim simlices & co-dim-1 simplices.

We take the best path. If the minimal simplex has codim ≥ 2, we bypass it & use its star relying on the lemma.

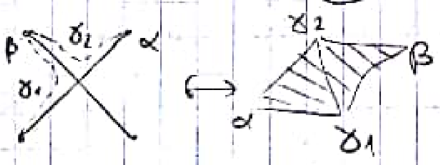
<u>Proof of thm.</u> :

Let $[\alpha], [\beta]$ vertices in $A(S,V)$

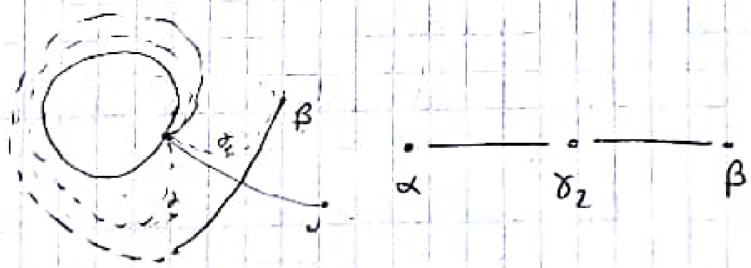
<u>case 0</u>: $|\alpha \cap \beta| = 0$  

<u>case 1</u>: $|\alpha \cap \beta| = 1$

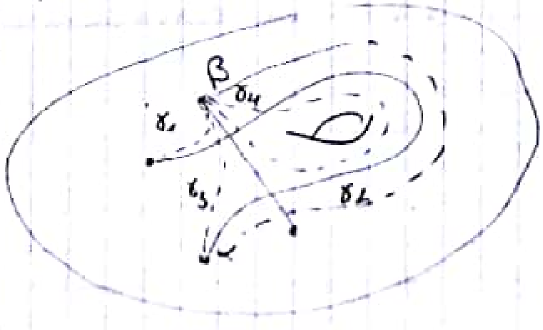
(problem, γ_1 or γ_2 can be isotopic to the boundary therefore we make the following assumption)

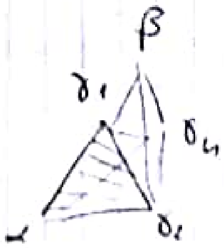


<u>Assume</u>: Exactly one vertix of V in each Component of ∂S



<u>case 2</u>: $|\alpha \cap \beta| = 2$





The proof of the thm will follow from:

Prop. 1:

If there is exaclty one vertex of V in every component of ∂S & $A(S,V) \neq \emptyset$ then $A(S,V)$ is contractible.

Prop. 2:

If V' is obtained from V by adding v' to some component of ∂S already containing a pt, $v \in V$, then $A(S,V')$ is homeomorphic to the suspension $\Sigma A(S,V)$ of $A(S,V)$

$$\Sigma X = I \times X \Big/ \begin{matrix} (x_1,0) \sim (x_2,0) \\ (x_1,1) \sim (x_2,1) \\ \forall x_1, x_2 \in X \end{matrix}$$