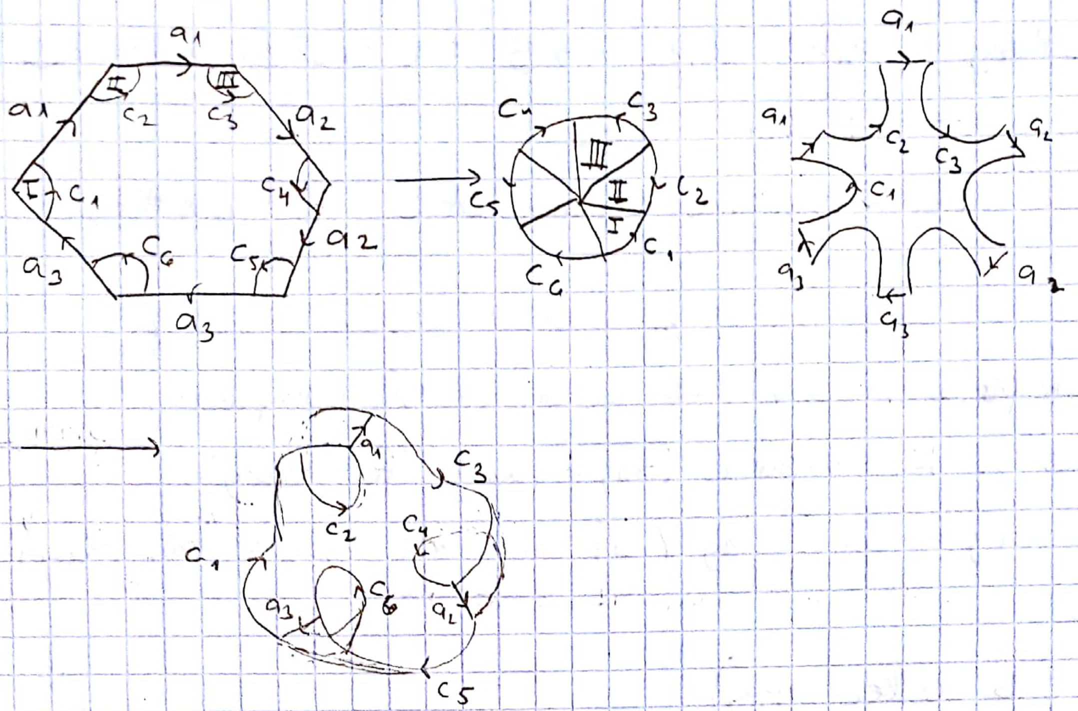
Lecture 4:

12/3/2017



The arc complex:

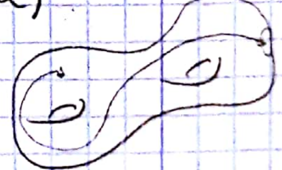
S: Compact (connected) surface, possibly with boundary

V: non empty finite set of vertices $V=\{v_1,\dots,v_n\} \subseteq S$

(Assume: at least one vertex from V in every boundary component

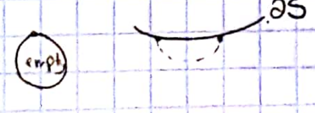
arc:

An embedding of $[0,1]$ in S, with end points in V, but interior disjoint from V & ∂S (boundary) (endpoints may coincide)



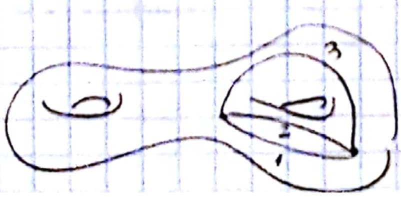
Essential arc:

not allowed:



does not cut S into two connected components one of which is a disc meeting V only at endpoints of the arc.

(*) An arc-System is a collection $\{\alpha_1,\dots,\alpha_k\}$ of essential arcs in S which are disjoint except perhaps the endpoints of the arc, & such that no two are "ambient isotopic" fixing V.



can't have 1 & 2 but can have 1 & 3.

Def:

Ambient isotopy: $g, h : [0,1] \longrightarrow S$ are two embeddings.

$F : S \times [0,1] \longrightarrow S$ continuous

$\forall\, t \in [0,1] :$ F_t : homeomorphism of S. F_0 = identity
$F_t \circ g$ $F_1 \circ g$

Def:

The arc-complex $A(S,V)$ is a simplical complex (קבוצה של קבוצות שכל תת קבוצה ~~מוכלת~~ שייכת) which has a k-simplex $[\alpha_0, \dots, \alpha_k]$ for every ambient-isotopy class fixing V of an arc system $\{\alpha_0, \dots, \alpha_{k+1}\}$ of $k+1$ arcs.

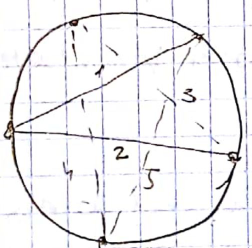
(faces: sub collections)

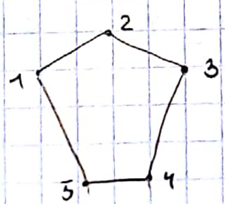
Examples:

Disc, with 3 points on boundary:  $A(S,V) = \emptyset$

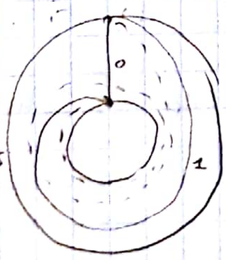
הקשתות הופכות לנקודות ומחברים 2 אם הם יכולים להיות ביחד

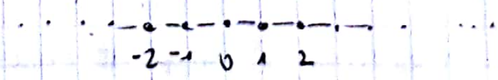
$A(S,V) = \underset{1}{\cdot} \quad \underset{2}{\cdot}$

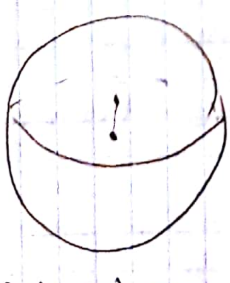


 $\leftarrow A(S,V)$

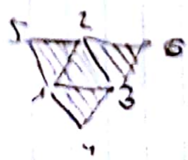
$S^2 \setminus 2$

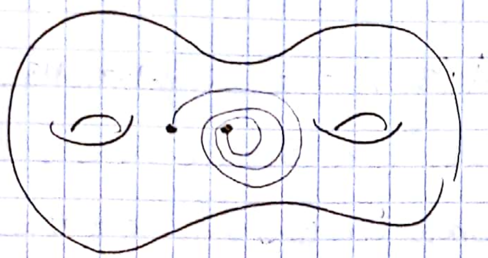


$A(S,V)$: 



$A(S,V)$: $\cdot$

$A(S,V)$: 



$A(S,V) = ??$

(*) The top-dimensional simplices in $A(S,V)$ correspond to triangulations of S. (two vertices / two edges of a triangle may coincide)

what is the dimension of a simplex?

(*) All maximal simplices of $A(S,V)$ have the same dimension

pf:

Let n be the number of triangles in a triangulation of S closed surface.

$$\chi(S) = |V| - \frac{3n}{2} + n = |V| - \frac{n}{2}$$

קודקודים   צלעות   פאות

← why does this prove the statement?

$$\Longrightarrow \quad n = 2(|V| - \chi(S))$$

for S with boundary - a similar computation shows that the number of arcs in a triangulation is fixed.

Thm (Harer, 85'):

$A(S,V)$ is either: 1) empty

or

2) homeomorphic to a sphere, if S is a disc & $V \subseteq \partial S$

or

3) Contractible (otherwise)

Hatcher, 1991: Much simpler proof. "Triangulation of Surfaces".

Corollary:

Every two triangulations of (S,V) are connected by elementary moves:

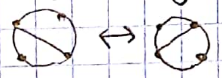
$\square$

Pf of Corollary:

top dimensional simplices of $A(S,V)$ = triangulations.

codim-1 simplices = triangulations \ a single arc. Here there are two options: 1) The arc belongs to two different triangles
2) Belongs to one triangle.

need to show:

Any two top-dimensional simplices in $A(S,V)$ can be joined by a path passing only through open codim-1/top-dim faces.
(Unless A consists of disjoint points).
this happens when: 

By Thm: there is a path, but may pass through small-dimesional faces.

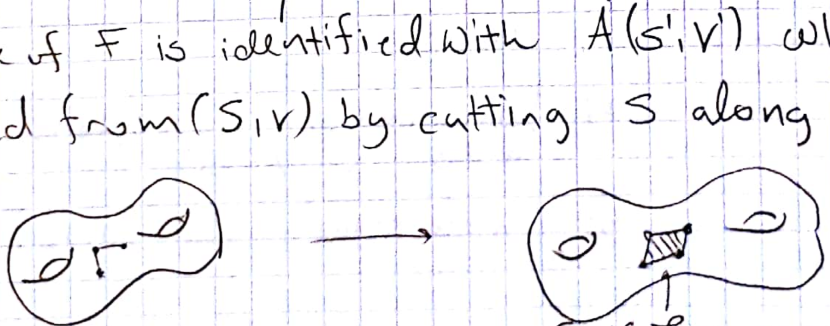
By induction on $\dim A$:

- if $\dim A = 1$ ✓ by thm.
- if $\dim A \geq 2$, say the path passes through some face F of codim ≥ 2. We show that the link of F is connected.
because then we can construct a path going through faces of higher codim by induction

~~A link~~

The link of $F = \{ E \text{ face of } A \mid E \cap F = \emptyset, E \cup F \in A \}$

The link of F is identified with $A(S',V')$ where (S',V') is obtained from (S,V) by cutting S along the arcs of F.



$$\underbrace{\dim A(S',V')}_{= \text{codimension of } F - 1 \geq 1} < \dim A(S,V)$$

$\Rightarrow \dim A(S',V')$ is connected.