Lecture 2 5/3/2017

Classification of surfaces:

Thm. [Dehn-Heegard 1907]:

The following is the complete list of connected, closed surfaces (up to homeo):

1) S^2

2) $\underbrace{T \# T \# \dots \# T}_{\text{times } n}$ $\forall n \geq 1$

3) $\underbrace{P \# \dots \# P}_{n \text{ times}}$ $\forall n \geq 1$ $(P = \mathbb{R}P^2)$

Proof:

Part 1: every 2 in the list are not homeomorphic.

Argument 1: We saw:

$$\left.\begin{aligned} \chi(S^2) &= 2 \\ \chi(\underbrace{T \# \dots \# T}_{n \text{ times}}) &= 2 - 2n \end{aligned}\right\} \text{orientable}$$

$$\chi(\underbrace{P \# \dots \# P}_{n \text{ times}}) = 2 - n \quad - \text{ non-orientable}$$

Orientability:

A surface is orientable if it does not contain an embedded Möbius band.

$\iff$ $\exists$ continuous choice of a normal/orientation at every point.

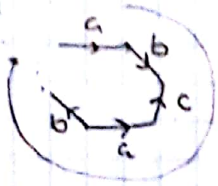
Fact:

Euler constant & orientability are topological invariants.

$\Rightarrow$ Every 2 in the list are not homeomorphic.

Argument 2: using fundamental group.

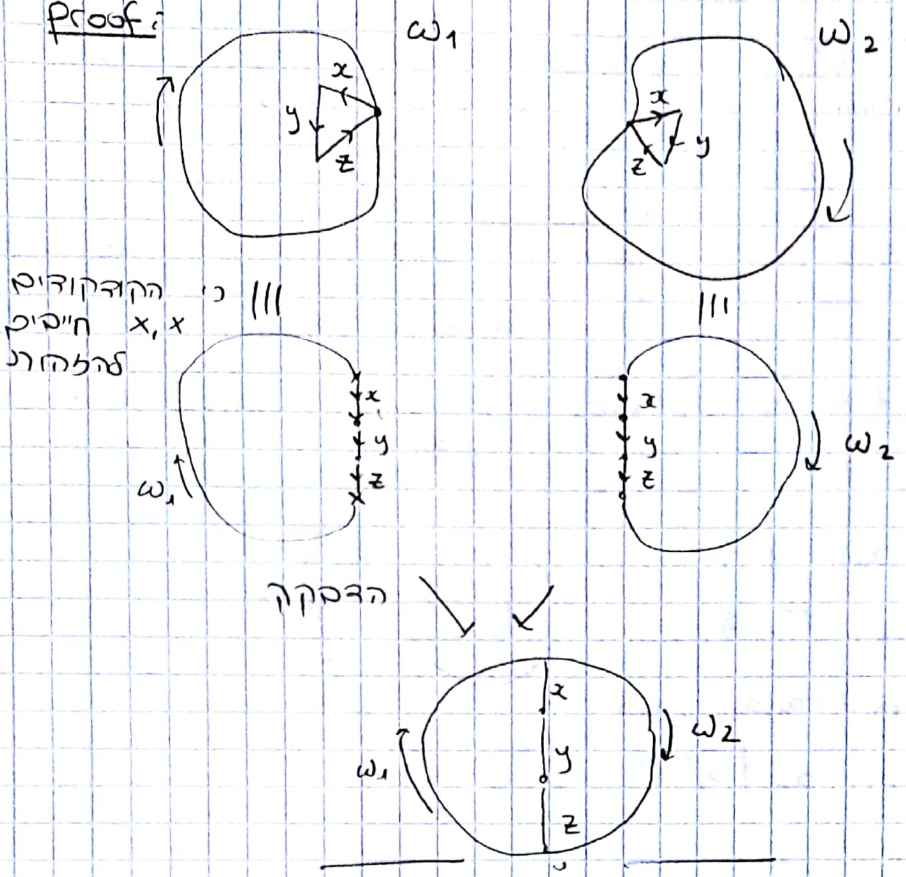
Canonical representation of these surfaces

Assume that S_1 represented by :  $w_1 = abc^{-1}a^{-1}b\dots$

Assume that S_2 represented by w_2

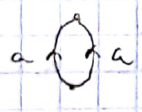
& w_1, w_2 are disjoint (אין אותיות זהות בשני המילים) then $S_1 \# S_2$ is represented by $\omega_1\omega_2$

Proof:

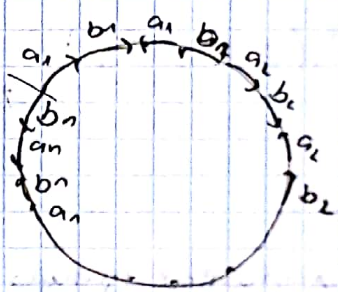


Canonical representations:

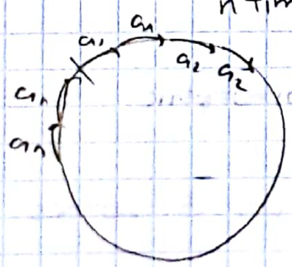
S^2 :



$\underbrace{T\#\ldots\#T}_{n \text{ times}}$:



$\underbrace{P\#\ldots\#P}_{n \text{ times}}$.



$T\#P = ?$

$\chi(T\#P) = \chi(T) + \chi(P) - 2 = -1$

orientable? no. $\Longrightarrow$ $T\#P = P\#P\#P$

Thm. (Seifert - Van Kampen):

Let (X, x_0) be a pointed, path-connected top space. Assume $X = A\cup B$ with A, B open, $x_0 \in A, B$, $A\cap B$ path-connected. Then:

$$\pi_1(X, x_0) = \frac{\pi_1(A, x_0) * \pi_1(B, x_0)}{\pi_1(A\cap B, x_0)} = \frac{\langle \pi_1(A, x_0), \pi_1(B, x_0)\rangle}{\langle\langle i_*^{A\cap B\to A}(\gamma) = i_*^{A\cap B\to B}(\gamma)\rangle\rangle \ \forall \gamma \in \pi_1(A\cap B)}$$

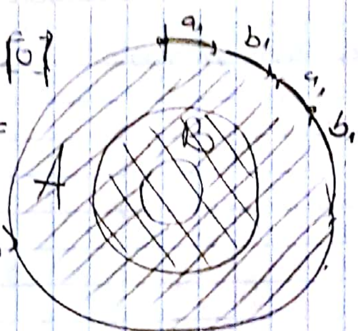
$\pi_1(S^2) = \{0\}$

$\pi_1(\underbrace{T\#\ldots\#T}_{n \text{ times.}}) = ?$

$\pi_1(B, x_0) = \{0\}$

$\pi_1(A, x_0) = \pi_1(\text{∞}\cdot) =$
2n עלים

$= F_{2n} = \langle a_1, b_1, \ldots, a_n b_n \rangle$



החבורה החופשית על 2n יוצרים

$$\pi_1(A \cap B, x_0) \cong \mathbb{Z} = \langle \gamma \rangle$$

$i_*^{A\cap B \to B}(\gamma) = e$, $i_*^{A \cap B \to A}(\gamma) = \underline{a_1 b_1 a_1^{-1} b_1^{-1}} \cdots \underline{a_n b_n a_n^{-1} b_n^{-1}}$

$= [a_1, b_1] \cdots [a_n, b_n]$

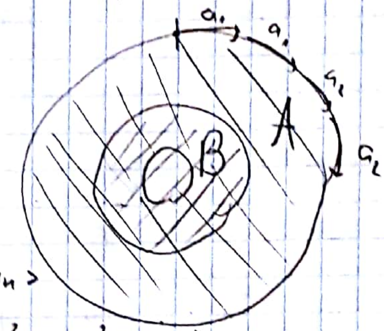
$\Longrightarrow \pi_1(T\#\ldots\#T) = \langle a_1, b_1, \ldots, a_n, b_n \mid [a_1, b_1] \cdots [a_n, b_n] = e \rangle$

$\pi_1(P\#\ldots\#P) = ?$

$\pi_1(B, x_0) = \{0\}$

$\pi_1(A, x_0) = \pi_1(\text{∞}, \cdot) =$

$= F_n = \langle a_1, a_2, \ldots, a_n \rangle$



$\pi_1(P\#\ldots\#P) = \langle a_1, a_2, \ldots, a_n \mid a_1^2 a_2^2 \ldots a_n^2 = e \rangle$

$\mathrm{Abel}(\pi_1(T\#\ldots\#T)) = \mathbb{Z}^{2n}$

$\mathrm{Abel}(\pi_1(P\#\ldots\#P)) = \underset{a_1\mathbb{Z} + a_2\mathbb{Z} + \ldots + a_n\mathbb{Z}}{\mathbb{Z}^n} / \langle 2a_1 + 2a_2 + \ldots + 2a_n = 0 \rangle = \mathbb{Z}^{n-1} \times \mathbb{Z}_2$

$a_1, a_2, \ldots, a_{n-1}, a_1 + a_2 + \ldots + a_n$

Corollary:

The fundamental groups are not isomorphic.

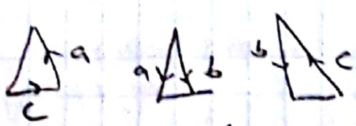
Part II of proof: Why every surface is homeo. to one of the surfaces in the list.

Thm. (Triangulation, Radó 1925):

Every surface can be triangulated.

(a 2-dim simplical complex)

I.e. every surface can be obtained as a finite union of disjoint triangles with edges identified in pairs.



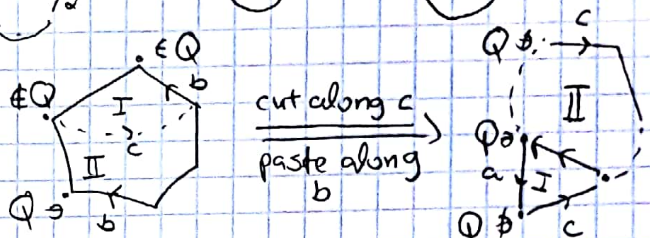
$\Rightarrow$ Every surface can be obtained as a polygon with 2n edges, that are glued in pairs.

Step 1: Reduction to a single vertex (or to $a^{-1}a$)

If there is more then one equivalence class of vertices: let Q one of these classes.

- If $|Q| = 1$

- If $|Q| \geq 2$



this reduces $|Q|$ by 1