Lecture 7:

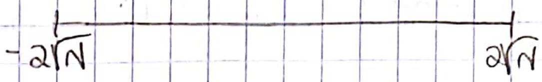
5/12/2017

Reminder:

Gaussian Unitary Ensemble (GUE)

$(\xi_{ij}), (\eta_{ij})$ IID $N(0,1)$

$$H_N = \begin{pmatrix} \xi_{11} & \frac{\xi_{2i}+i\eta_{2i}}{\sqrt{2}} & \\ & \ddots & \\ \text{Herm} & & \xi_{NN} \end{pmatrix}$$



Eigenvalues $\lambda_1^N \leq \ldots \leq \lambda_N^N$ are indep. of the eigenvectors.

$H_N \overset{d}{=} U H_N U^{-1}$ $\forall U$ unitary.

Thm.:

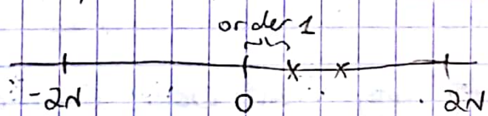
Joint density of $(\lambda_1^N, \ldots, \lambda_N^N) \in \mathbb{R}^N$ w.r.t Leb. measure on $\mathbb{R}^N$ is:

$$N!\, C_N\, \mathbb{1}_{x_1 \leq \ldots \leq x_N} |\Delta(x)|^2 \prod_{i=1}^N e^{-\frac{x_i^2}{2}} \quad ; \quad \Delta(x) = \prod_{i<j}(x_j - x_i)$$

$$C_N = \frac{1}{(\sqrt{2\pi})^N \prod_{j=0}^{N-1} j!} \cdot \frac{1}{N!}$$

Goal: (1) spacing dist, dist of the (2) maximal eigenvalues.

$\sqrt{N}(\lambda_1^N, \ldots, \lambda_N^N)$



(spacing dist.)

$\frac{1}{\sqrt{N}}(\lambda_1^N, \ldots, \lambda_N^N)$

For (1), need to find $\lim_{N\to\infty} P\left(\sqrt{N}\lambda_1^N, \ldots \sqrt{N}\lambda_N^N \notin \left[-\frac{t}{2}, \frac{t}{2}\right]\right)$

Main tool: $(\lambda_1^N, \ldots, \lambda_N^N)$ form a determinantal Point Process.
Probability of interest is a Fredholm determinant.

$\mu_1, \ldots, \mu_N$ are the unordered eigenvalues obtained by uniformly permuting $\lambda_1^N, \ldots, \lambda_N^N$. Joint density of $(\mu_1^N, \ldots, \mu_N^N)$ is:

$$C_N |\Delta(x)|^2 \prod_{i=1}^N e^{-\frac{x_i^2}{2}}$$

Hermite Polynomials:

These are the orthogonal Poly. of the Gaussian density $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. That is, in $L_2(\mathbb{R}, \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}})$ Consider $1, x, x^2, \dots$ & apply Gram-Schmidt Procedure to get (normalized) Hermite Poly. In particular, $h_n(x)$ is a Poly. of degree n.

orthogonality: $$\int h_k(x)\, h_l(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 0 \quad \text{when } k \neq l.$$

Def.:

The n'th Hermite Poly. is:
$$h_n(x) := (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} \quad \text{(Rodriguez-type formula)}$$

Properties:

(1) $h_0 \equiv 1$, $h_1(x) = x$. For $n \geq 0$. $\overset{(2)}{h_{n+1}(x) = x h_n(x) - h_n'(x)}$ $(*)$

$\Longrightarrow$ h_n is a monic poly. of deg n.

coefficient of x^n is 1.

For $(*)$: $h_{n+1}(x) = \left((-1)^n e^{-\frac{x^2}{2}} h_n(x)\right)' \cdot (-1)^n e^{\frac{x^2}{2}}$

(3) also, h_n is even or odd according to the parity of n.

(4) $\forall k, l \geq 0$, $\int h_k(x) h_l(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = k!\, \delta_{kl}$

$$\overset{=}{(-1)^l \int h_k(x) \frac{d^l}{dx^l} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} dx} \underset{\uparrow}{=} \int \frac{d^l}{dx^l} h_k(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

integration by parts.

If $l > k$ this is zero. For $k = l$, get $k!$ since h_k is monic.

Define also the oscillator wave function:
$$\Psi_n(x) = \frac{h_n(x) e^{-\frac{x^2}{4}}}{(\sqrt{2\pi}\, n!)^{1/2}}$$

then $\int \Psi_k(x) \Psi_l(x) dx = \delta_{kl}$

Lemma 3.2.2:

For any $1 \le p \le N$, the joint density of $\mu_1, \dots, \mu_p$ is:

$$\frac{(N-p)!}{N!} \det_{i,j=1}^{p} K^{(N)}(x_i, x_j)$$

where $K^N(x,y) = \sum_{k=0}^{N-1} \Psi_k(x)\Psi_k(y)$

$$\begin{pmatrix} K^N(x_1,x_1), & K^N(x_1,x_2) & \dots \\ K^N(x_2,x_1) & & \\ \vdots & \ddots & \end{pmatrix}$$

Proof:

Start with $p = N$. $\Delta(x) = \det \begin{pmatrix} 1 & \dots & 1 \\ x_1 & & x_N \\ \vdots & & \\ x_1^{N-1} & \dots & x_N^{N-1} \end{pmatrix} = \det \begin{pmatrix} 1 & \dots & 1 \\ h_1(x_1) & & h_1(x_N) \\ \vdots & & \vdots \\ h_{N-1}(x_1) & & h_{N-1}(x_N) \end{pmatrix}$

as h_n is a monic poly. of deg. n.

Then $C_N |\Delta x|^2 \prod_{i=1}^{N} e^{-\frac{x^2}{2}} =$

$$= C_N \det \begin{pmatrix} 1 & \dots & 1 \\ h_1(x_1) & & h_1(x_N) \\ \vdots & & \\ h_N(x_1) & & h_N(x_N) \end{pmatrix}^2 \det \begin{pmatrix} e^{-\frac{x^2}{4}} & & 0 \\ & \ddots & \\ 0 & & e^{-\frac{x^2}{4}} \end{pmatrix}^2 =$$

$$= \underbrace{C_N (\sqrt{2\pi})^N \prod_{j=0}^{N-1} j!}_{= 1/N!} \det \begin{pmatrix} \Psi_0(x_1) & \dots & \Psi_0(x_N) \\ \Psi_1(x_1) & & \Psi_1(x_N) \\ \vdots & & \vdots \\ \Psi_N(x_1) & \dots & \Psi_N(x_N) \end{pmatrix}^2 = \frac{1}{N!} \det \begin{pmatrix} K^{(N)}(x_1,x_1) & K^N(x_1,x_2) & \dots \\ K^{(N)}(x_2,x_1) & & \\ \vdots & & \end{pmatrix}$$

multiply transpose of matrix by itself.

How to integrate this density?

Lemma 3.2.3:

Let $f_1, \dots, f_n, g_1, \dots, g_n \in L_2(\mathbb{R})$,

$$\frac{1}{n!} \iiint \dots \int \det \begin{pmatrix} \sum_{k=1}^{n} f_k(x_1) g_k(x_1) & \sum_{k=1}^{n} f_k(x_1) g_k(x_2) & \dots \\ \sum_{k=1}^{n} f_k(x_2) g_k(x_1) & & \\ \vdots & & \end{pmatrix} dx_1 \dots dx_n =$$

$$\circleddash\ \frac{1}{n!}\int\cdots\int \det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix} \det\begin{pmatrix} g_1(x_1) & \cdots & g_1(x_n) \\ \vdots & \ddots & \vdots \\ g_n(x_1) & \cdots & g_n(x_n)\end{pmatrix} dx_1 \dots dx_n$$

$$= \det\begin{pmatrix} \int f_1(x) g_1(x) & \int f_1(x) g_2(x) & \cdots \\ \int f_2(x) g_1(x) & & \\ \vdots & & \ddots \end{pmatrix}$$

Proof:

$\mu = \sigma \circ \tau^{-1}$

$$= \frac{1}{n!}\sum_{\sigma,\tau\in S_n} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \underbrace{\int\cdots\int \prod_{i=1}^{n} f_{\sigma(i)}(x_i)\, g_{\tau(i)}(x_i)\, dx_i}_{\prod_{i=1}^{N}\int f_{\sigma(i)}(x)\, g_{\tau(i)}(x)\, dx} =$$

$$= \sum_{\mu\in S_N} \operatorname{sgn}(\mu) \prod_{i=1}^{N} f_{\mu(i)}(x)\, g_i(x)\, dx = \text{wanted det.}$$

In part. $\iint\cdots\int \frac{1}{N!} \det_{i,j=1}^{N} K^N(x_i, x_j)\, dx_1 \dots dx_N = 1.$

proceed to $1 \le p < N$. Need to evaluate:

$$\iint\cdots\int \frac{1}{N!} \det\begin{pmatrix} \psi_0(x_1) & \cdots & \psi_0(x_n) \\ \psi_1(x_1) & & \\ \vdots & & \end{pmatrix}^2 \prod_{i=p+1}^{N} dx_i =$$

open det. using permutations.

$$= \frac{1}{n!} \sum_{\sigma,\tau\in S_N} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \int\cdots\int \prod_{i=0}^{N-1} \psi_{\sigma(i)}(x_{i+1})\,\psi_{\tau(i)}(x_{i+1}) \prod_{i=p+1}^{N} dx_i =$$

perm. of $\{0,\dots,N-1\}$

$$= \frac{1}{n!} \sum_{\substack{\sigma,\tau\in S_N \\ \sigma(i)=\tau(i),\ i>p-1}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \prod_{i=0}^{p-1} \psi_{\sigma(i)}(x_{i+1})\,\psi_{\tau(i)}(x_{i+1}) =$$

$$= \frac{1}{N!} \sum_{0\le\nu_1<\nu_2<\dots<\nu_p\le N-1}\ \sum_{\substack{\sigma,\tau\in S_N \\ \sigma(i)=\tau(i)\ i>p-1 \\ \{\sigma(1),\dots,\sigma(p)\}=\{\nu_1,\dots,\nu_p\}}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \prod_{i=0}^{p-1} \psi_{\sigma(i)}(x_{i+1})\,\psi_{\tau(i)}(x_{i+1})$$

$$\cancel{= \frac{1}{N!} \sum_{1\le\nu_1<\nu_2<\dots<\nu_p\le N-1} (N-p)! \sum_{\mu:\{0,\dots,p-1\}\to\{\nu_1,\dots,\nu_p\}} \operatorname{sgn}(\mu) \prod_{i=0}^{p-1} \psi_{\mu(i)}(x_{i+1})\,\psi_{\nu_i}(x_{i+1})}$$

$\mu = \sigma\circ\tau^{-1}$

$$= \frac{(N-p)!}{N!} \sum_{0 \le \nu_1 < \dots < \nu_p \le N-1} \det \begin{pmatrix} \psi_{\nu_1}(x_1) & \psi_{\nu_1}(x_2) & \dots & \psi_{\nu_1}(x_p) \\ \psi_{\nu_2}(x_1) & & & \vdots \\ \vdots & & & \\ \psi_{\nu_p}(x_1) & \dots & & \end{pmatrix}^2 =$$

$$= \frac{(N-p)!}{N!} \det \left(\begin{pmatrix} \psi_1(x_1) & \psi_2(x_1) & \dots & \psi_{N-1}(x_1) \\ \vdots & & & \\ \psi_1(x_p) & \dots & & \psi_{N-1}(x_p) \end{pmatrix} \begin{pmatrix} \psi_1(x_1) & \dots & \psi_1(x_p) \\ \vdots & & \vdots \\ \psi_{N-1}(x_1) & \dots & \psi_{N-1}(x_p) \end{pmatrix} \right)$$

Cauchy-Binet:

$$p\Big(\ \overbrace{\quad A \quad}^{N}\ \Big)\begin{pmatrix} B \end{pmatrix} = \underbrace{\begin{pmatrix} C \end{pmatrix}}_{p} p, \qquad B: N \times p$$

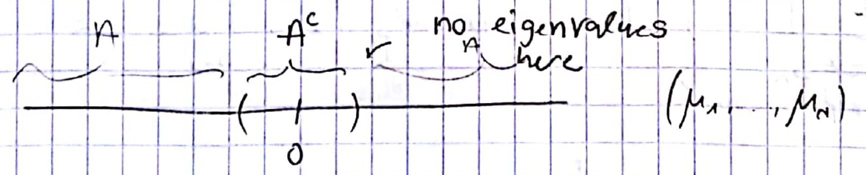
$$\det(C) = \sum_{\substack{I \subseteq \{1,\dots,N\} \\ |I| = p}} \det(A_I)\det(B_I)$$

A_I - columns of A indexed by I
B_I - rows of B indexed by I

$$= \frac{(N-p)!}{N!} \det_{i,j=1}^{p} K^N(x_i, x_j)$$

$$\Downarrow$$

Joint density of $\mu_1, \dots, \mu_p$ is $\frac{(N-p)!}{N!} \det_{i,j=1}^{p} \left(K^N(x_i, x_j)\right)$



Lemma 3.2.4:

For any Borel $A \subseteq \mathbb{R}$,

$$P(X_1^N \in A, \dots, X_N^N \in A) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \underbrace{\int_{A^c}\int_{A^c} \dots \int_{A^c}}_{k\text{-fold integral}} \overbrace{\det_{i,j=1}^{k} K^N(x_i, x_j)}^{k\text{-fold det.}} \prod_{i=1}^{k} dx_i$$

(same as summing $1 \le k \le N$ as the expr is 0 afterward)

This is a <u>Fredholm determinant</u> (of $K^{(N)}$ w.r.t Leb. meas. on A^c)

<u>Proof</u>:

Probability in question equals $\frac{1}{N!}\int_A \cdots \int_A \det\left(K^N(x_i,x_j)\right)\prod_{i=1}^N dx_i$ =

Lemma 3.2.3 with $f_i = g_i = \psi_i \mathbb{1}_A$

$$\overset{\downarrow}{=} \det\begin{pmatrix} \int_A \psi_1(x)\psi_1(x)dx & \int_A \psi_1(x)\psi_2(x)dx & \cdots \\ \int_A \psi_2(x)\psi_1(x)dx & & \\ \vdots & & \end{pmatrix} =$$

$$= \det\left(I - \begin{pmatrix} \int_{A^c} \psi_1(x)\psi_1(x)dx & \int_{A^c}\psi_1(x)\psi_2(x)dx & \cdots \\ \int_{A^c} \psi_2(x)\psi_1(x)dx & & \\ \vdots & & \end{pmatrix}\right) \textcircled{=}$$

$$\det(A+B) = \sum_{I\subseteq\{1,\dots,N\}} \det\left((A|B)_I\right)$$

I'th rows taken from A
I^c'th rows taken from B

$$\textcircled{=}\ 1 + \sum_{k=1}^N \sum_{\substack{J\subseteq\{1,\dots,N\}\\ |J|=k}} (-1)^k \det\begin{pmatrix} \int_{A^c}\psi_{J_1}(x)\psi_{J_1}(x)dx & \int_{A^c}\psi_{J_1}(x)\psi_{J_2}(x)dx & \cdots \\ \vdots & & \\ \vdots & & \int_{A^c}\psi_{J_k}(x)\psi_{J_k}(x) \end{pmatrix}$$

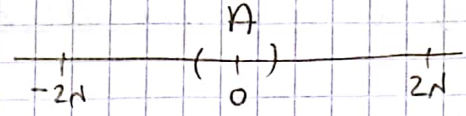
rows taken from second term
$J = \{J_1,\dots,J_k\}$, $J_1 < J_2 < \dots < J_k$

$$= 1 + \sum_{k=1}^N \sum_{1\le J_1 \le \dots \le J_k \le N} \frac{(-1)^k}{k!}\int_{A^c}\int_{A^c} \det\begin{pmatrix} \psi_{J_1}(x_1) & \psi_{J_1}(x_2) & \cdots \\ \psi_{J_2}(x_1) & & \vdots \\ \vdots & \cdots & \psi_{J_k}(x_k)\end{pmatrix}^2 \prod_{i=1}^k dx_i \underset{\substack{\uparrow\\ \text{Cauchy-}\\ \text{Binet}}}{=}$$

$$= 1 + \sum_{k=1}^N \frac{(-1)^k}{k!} \int_{A^c}\cdots\int_{A^c} \det \underbrace{\begin{pmatrix} \psi_1(x_1) & \cdots & \psi_{N-1}(x_1) \\ \psi_1(x_k) & \cdots & \psi_{N-1}(x_k)\end{pmatrix}\begin{pmatrix}\text{same} \\ \text{transpose}\end{pmatrix}}_{= K^{(N)}(x_i,x_j)} \prod_{i=1}^k dx_i =$$

$= \text{Same with } \sum_{k=1}^{\infty}$ as the rank of the product of matrices is $\le N$.

Spacing dist.:



Thm. (Gaudin - Mehta):

For any compact $A \subseteq \mathbb{R}$, $\lim_{N \to \infty} P(\sqrt{N} X_1^N, \dots, \sqrt{N} X_N^N \notin A) =$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_A \cdots \int_A \det_{i,j=1}^{k} \left(S(x_i, x_j) \right) \prod_{i=1}^{k} dx_i$$

where S is the Sine Kernel:

$$S(x,y) = \begin{cases} \frac{1}{\pi} \frac{\sin(x-y)}{x-y} & x \neq y \\ \frac{1}{\pi} & x = y \end{cases}$$

First step in proof:

Define "approximation of sine kernel",

$$S^{(N)}(x,y) = \frac{1}{\sqrt{N}} K^{(N)}\left(\frac{x}{\sqrt{N}}, \frac{y}{\sqrt{N}} \right)$$

Main Lemma:

$\lim_{N \to \infty} S^N(x,y) = S(x,y)$ uniformly for x,y in a compact subset A of $\mathbb{R}$.

Proof of thm. with main lemma & continuity:

$$P(\sqrt{N} X_1^N, \dots, \sqrt{N} X_N^N \notin A) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{A/\sqrt{N}} \cdots \int_{A/\sqrt{N}} \det_{i,j} K^N(x_i, x_j) \prod_{i=1}^{k} dx_i$$

$$\underset{\text{divide each } x_i \text{ by } \sqrt{N}}{=} 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_A \cdots \int_A \det_{i,j=1}^{k} S^{(N)}(x_i, x_j) \prod_{i=1}^{k} dx_i \xrightarrow[\text{continuity}]{} \text{same with } S \text{ replacing } S^{(N)}$$