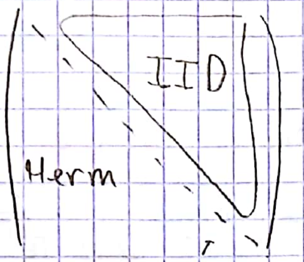
Lecture 6: 28/11/2017

Gaussian Orthogonal/Unitary ensemble (GOE, GUE):

real symmetric Hermition




IID on diagonal.

start with $(\xi_{i,j})$, $(\eta_{i,j})$ IID $N(0,1)$
indep.

GOE: $\overset{N(0,2)}{\left(\begin{matrix} \sqrt{2}\,\xi_{11} & \xi_{12} \\ \xi_{12} & \sqrt{2}\,\xi_{22} \end{matrix}\right)}$ $(n=2)$, $\left(\begin{matrix} \sqrt{2}\,\xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{12} & \sqrt{2}\,\xi_{22} & \xi_{23} \\ \xi_{13} & \xi_{2,3} & \sqrt{2}\,\xi_{33} \end{matrix}\right)$ $(n=3)$

$N(0,1)$ $\quad N_{\mathbb{C}}(0,1)$

GUE: $\left(\begin{matrix} \xi_{11} & \frac{\xi_{12}+i\eta_{12}}{\sqrt{2}} \\ \frac{\xi_{12}-i\eta_{12}}{\sqrt{2}} & \xi_{22} \end{matrix}\right)$ $(N=2)$ $\qquad \mathbb{E}\,|N_{\mathbb{C}}(0,1)|^2 = 1$

Main feature: EigenValues & eigenVectors are indep.

Density of GOE/GUE on matrices:

Reference measure:

Denote by $H_N^{(\beta)}$ the set of:

- $\beta=1$: real symmetric matrices
- $\beta=2$: Complex Hermition matrices.

Denote by $l_N^{(\beta)}$ the "Lebesgue" measure on $H_N^{(\beta)}$ defined by:

- $\beta=1$: Product Lebesgue on diagonal & above diagonal entries.
- $\beta=2$: Product Lebesgue on diagonal & on real & imaginary parts of each above diag. entry.

$\dim(l_N^{(1)}) = \binom{N+1}{2}$, $\dim(l_N^{(2)}) = N^2$.

For GOE, joint density is:

$$\prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{X_{i,i}^2}{4}} \prod_{j>i} \frac{1}{\sqrt{2\pi}} e^{-\frac{X_{i,j}^2}{2}} \, d\mu_N^{(1)} = \frac{1}{(2\pi)^{\frac{N(N+1)}{4}} \cdot 2^{N/2}} \underbrace{e^{-\frac{1}{4}\sum_{i,j} X_{i,j}^2}}_{= e^{-\frac{1}{4}\mathrm{Tr}(X^2)}}$$

Density of $X$ is a function only of the eigenvalues of $X$. For GUE, joint density $= \frac{1}{2^{N/2}} \frac{1}{\pi^{N^2/2}} e^{-\frac{1}{2}\mathrm{tr}(X^2)}$.

Can write $X = UDU^*$

- $U$: $\beta=1$: Orthogonal; $\beta=2$: Unitary. Columns are the eigenvectors.
- $D$: diagonal, eigenvalues on diagonal

"When you condition on the eigenvalues, the eigenvectors are a uniformly picked (Haar measure) orthogonormal basis."

Main theorem of today: Joint density of the eigenvalues $\lambda_1 \le \lambda_2 \le \dots \le \lambda_N$

Def:

For $x \in \mathbb{C}^N$, $\Delta(x) = \prod_{i<j}(x_i - x_j) = \det \begin{pmatrix} 1 & 1 & & 1 \\ x_1 & x_2 & & x_N \\ \vdots & \vdots & & \vdots \\ x_1^{N-1} & x_2^{N-1} & & x_N^{N-1} \end{pmatrix}$

Thm.:

Let $X$ be GOE / GUE ($\beta=1$ / $\beta=2$) distributed. The joint dist. of $(\lambda_1, \lambda_2, \dots, \lambda_N)$ is a density w.r.t. Leb. meas. on $\mathbb{R}^N$, given by:

$$\underbrace{N!\, \bar{G}_N^{(\beta)}} |\Delta(x)|^\beta \prod_{i=1}^{N} e^{-\frac{\beta x_i^2}{4}} \cdot \mathbb{1}_{x_1 \le x_2 \le \dots \le x_N} \prod_{i=1}^{N} dx_i$$

Selberg integral

$$= \frac{1}{(2\pi)^{N/2}} \left(\frac{\beta}{2}\right)^{\frac{N}{2} + \frac{\beta}{2}\binom{N}{2}} \prod_{j=1}^{N} \frac{\Gamma(\beta/2)}{\Gamma(j\beta/2)}$$ where $\Gamma(s) := \int_0^\infty x^{s-1} e^{-x} dx$

Gamma function

→ $\Gamma(n+1) = n!$ for $n \ge 0$ integer.

→ $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

→ $\Gamma(x+1) = x\,\Gamma(x)$

Idea of Proof: $X = U D U^*$

orthogonal ($\beta=1$) / Unitary ($\beta=2$) eigenvectors — diagonal, eigenvalues

Study the map $(D,U) \longrightarrow UDU^*$, analyze what is the push-forward through it of a dist. on $D$.

Jacobian of the map will equal $G(U)\Delta(x)^\beta$ where $x$ is the diagonal of $D$.

This will not give us the value of $\bar{G}_N^{(\beta)}$. we will not calculate it now. Later, for $\beta=2$.

Example: $\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix}$ - orthogonal - 3 real degrees of freedom

More generally: - orthogonal - $\binom{N}{2}$ degrees of freedom
unitary - $2\binom{N}{2}$ degrees of freedom.

Proof:

Def. of desirable properties of $D, U$:

- $U$ is normalized if all diagonal entries are real & strictly positive.
- $U$ is good if it is normalized & has no zero entries.
- $U$ is very good if it is good & all its minors are non-zero

(minor $= \det(U_{I,J})$ where $I,J \subseteq \{1,\dots,N\}$, $|I| = |J| > 0$
$U_{I,J}$ - $I$'th rows of $U$, $J$'th columns of $U$)

- $D$ is distinct if all diagonal entries are distinct.
- $D$ is ordered if diagonal entries are decreasing (& distinct).
- $H_N^{(\beta)}$ - set of all $\beta=1$: real symmetric / $\beta=2$: Hermition matrices Endowed with Leb. measure $\ell_N^{(\beta)}$

- $M_N^{(\beta),dg}$ - $X \in M_N^{(\beta)}$, $X = UDU^*$, $D$-distinct, $U$-good

Lemma 1:
$\ell_N^{(\beta)}(M_N^{(\beta)} \setminus M_N^{(\beta),dg}) = 0$
Additionally, the map $(D, U) \longrightarrow UDU^*$ is one to one & onto from $(D^{do}, U^g)$ to $M_N^{(\beta),dg}$

distinct-ordered   good

(& $N!$ - to 1 if $D$ is not ordered)

Lemma 2:

distinct, very good

$$\ell_N^{(\beta)}(M_N^{(\beta)} \setminus M_N^{(\beta),dvg}) = 0$$

Lemma 3: ((smooth) parametrization of $U$):
The map $T: U^{(\beta),vg} \longrightarrow \mathbb{R}^{\beta\binom{N}{2}}$ given by:

$$T(U) = \left(\frac{U_{12}}{U_{11}}, \dots, \frac{U_{1N}}{U_{11}}; \frac{U_{23}}{U_{22}}, \dots, \frac{U_{2N}}{U_{22}}; \dots; \frac{U_{N-1,N}}{U_{N-1,N-1}}\right)$$

is one to one with smooth inverse.
Further, the set $T(U^{(\beta),vg})^c$ is closed with zero Lebesgue measure.

This gives a map from $\mathbb{R}^N \times \mathbb{R}^{\beta\binom{N}{2}}$ to $M_N^{(\beta),dvg}$ by:

$$(x, z) \xmapsto{\hat{T}} T^{-1}(z) D_x T^{-1}(z)^*$$

taking $x$ distinct & $z \in T(U^{(\beta),vg})$

diag values. $T(U)$ diagonal matrix with $x$ on diagonal

Proof of thm. from lemmas:
Analyze Jacobian of $\hat{T}$.

$$\begin{pmatrix} x_1 & 0 \\ 0 & x_N \end{pmatrix}$$

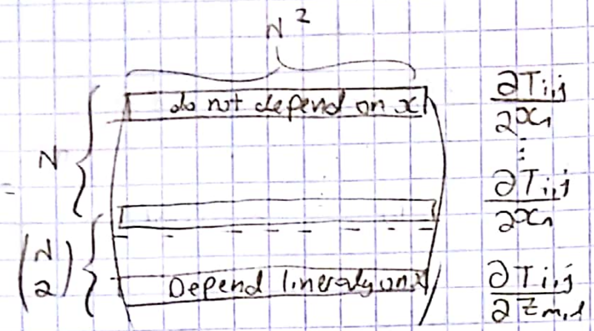
$J\hat{T} = \det\left(\frac{\partial \hat{T}_{ij}}{\partial x_k}, \frac{\partial \hat{T}_{ij}}{\partial z_{m,l}}\right)$. By def. $\left(\hat{T}_{ij} = T^{-1}(z) D_x T^{-1}(z)^*\right) = \sum_{k=1}^{N} C_k^{ij}(z) x_k$

is linear in $x$, with coefficients depending on $z$.
So, $\frac{\partial \hat{T}_{ij}}{\partial x_k}$ is a fcn only of $z$, $\frac{\partial \hat{T}_{ij}}{\partial z_{m,l}}$ is linear in $x$, with coef. depending on $z$.

Conclude that $\mathrm{Jac}(\hat{T})$ is a polynomial of $\deg \le \beta\binom{N}{2}$ in $x$, which is homegenious. (or the zero polynomial).

real case:

Jacobi matrix =




not justified & not used: $\mathrm{Jac}(\hat{T}) \neq 0$ on $H_N^{(\beta)\,\mathrm{dvg}}$ since $\hat{T}$ is invertible in a small neighborhood of each matrix there. (say, to ordered eigenvalues).

Implicit fcn. thm: $\mathrm{Jac}(\hat{T}) = 0$ if $x_i = x_j$ for some $i \neq j$ since $\hat{T}$ is non-invertible in a neigh of such points.

It follows that $\mathrm{Jac}(\hat{T})$ is a multiple of $\Delta(x)$ (by some poly. in $x$ with coef. depending on $z$).

In fact, $\mathrm{Jac}(\hat{T})$ is divisible by $\Delta(x)^\beta$. It follows, since $\mathrm{Jac}(\hat{T})$ is a poly. of deg $\beta\binom{N}{2}$ in $x$, that

$$\mathrm{Jac}(\hat{T}) = \Delta(x)^\beta \cdot C(z).$$

Now take $f$ a function of the eigenvalues of $X$.

$$E(f(X)) = \int f(\lambda_1, \dots \lambda_n) \text{ density of eigenvalues}.$$

$$E(f(X)) = \int_{H_N^{(\beta)}} f(\lambda_1(X), \dots, \lambda_N(X)) \cdot C_\beta e^{-\frac{\beta}{4}\operatorname{tr}(X^2)} \, dl_N^{(\beta)}(X)$$

$$= \int_{H_N^{(\beta)}} f(\lambda_1(X), \dots, \lambda_N(X)) \, C_\beta e^{-\frac{\beta}{4}\sum_{i=1}^N \lambda_i(X)^2} \, dl_N^{(\beta)}(X) \overset{\text{Jacobian Calculation}}{=}$$

$$= \iint_{x_1 \le \dots \le x_N} f(x_1, \dots, x_n) \, C_\beta e^{-\frac{\beta}{4}\sum_{i=1}^N x_i^2} \Delta(x)^\beta C(z) \, dx \, dz =$$

$$= \underbrace{\int C_\beta C(z) \, dz}_{\text{normalization const.}} \int_{x_1 \le \dots \le x_N} f(x_1, \dots, x_n) \underbrace{e^{-\frac{\beta}{4}\sum_{i=1}^N x_i^2} \Delta(x)^\beta \, dx}_{\text{density of eigenvalues}}$$

Why $\Delta(x)^2$ when $\beta = 2$ divides $\mathrm{Jac}(\hat{T})$ ?

For brevity, write $\hat{T}(x,z) = \underbrace{W D W^*}_{T^{-1}(z)}$

Differential:

$$d\hat{T} = (dW) D W^* + W (dD) W^* + W D (dW^*) \qquad (*)$$

$$\frac{\partial \hat{T}_{i,j}}{\partial z_{m,l}} = \left( \frac{\partial W}{\partial z_{m,l}} \cdot D W^* + \dots \right)_{i,j} \sim \text{explenation of } d\hat{T}.$$

Now, as $WW^* = I$, whence

$$0 = d(W^*W) = (dW^*) W + W^*(dW) \qquad (**)$$

$(*) + (**) \Longrightarrow$

$$W^* d\hat{T}\, W = W^*(dW) D + dD + D(dW^*) W \overset{(**)}{=}$$
$$= dD + W^*(dW) D - D W^*(dW)$$

Now suppose that $x_i = x_j$ for some $i \neq j$.

$$\underbrace{W^*(dW) D}_{\text{multiplying columns by entries of } D} - \underbrace{D\, W^*(dW)}_{\text{multiplying rows by entries of } D} = \begin{pmatrix} & \overset{i}{\underset{j}{0}} & \end{pmatrix}$$

Thus $\left( \frac{\partial \,\mathrm{Re}\, \hat{T}_{i,j}}{\partial z_{m,l}} \right)_{m,l} = \left( \frac{\partial \,\mathrm{Im}(\hat{T}_{i,j})}{\partial z_{m,l}} \right)_{m,l} \equiv 0$

## Proof of Lemma 1:

Claim:

Let $P$ be a poly. in $n$ variables. Then either $P$ is identically zero or $\underbrace{\{x \mid P(x) = 0\}}_{\text{closed}} = Z$ has zero Leb measure in $\mathbb{R}^n$.

## Idea of Proof:

Induction on $n$. $\mathrm{Leb}(Z) = \int dx_1 \int dx_2 \dots dx_n \, \mathbb{1}_Z(x_1, \dots, x_n) > 0$

$\vdots$

Distinct eigenvalues: The discriminant of the charectaristic polynomial of $X \in H_N^{(\beta)}$ is a poly. in the entries of $X$ which vanishes iff the char poly. had a double root, i.e., non distinct eignvalues.

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Goals:

Now suppose all eigenvalues of $X$ have multiplicity 1.

Let $\lambda$ be an eigenvalue of $X$.

Define $A = X - \lambda I$. Adjoint: $A^{adj}_{ij} = (-1)^{i+j} \det(A^{\hat{i},\hat{j}})$ erase i-th row & j-th column.

$\Longrightarrow A A^{(adj)} = \det(A) I = 0 \Longrightarrow$ Each column of $A^{(adj)}$ is a multiple of the unique eigenvector of $\lambda$.

$\Longrightarrow$ if all diagonal entries of $A^{(adj)}$ are non-zero, then the eigenvector of $\lambda$ has no zero entries.

Now, $A^{adj}_{i,i} = \det(A^{i,i}) = \det((X-\lambda I)^{i,i}) = \det(X^{i,i} - \lambda I)$

Conclusion: If there exists some eigenvalue $\lambda$ whose eigenvector has a zero entry then the char poly of $X$ & the char poly of $X^{i,i}$, for some $i$, have a common root.

Use that given two poly. $P, Q$, there exists another poly. $r(P,Q)$, the resultant of $P, Q$, whose coef are poly. in the coef of $P, Q$ which vanishes iff $P, Q$ have a common root.

∎

for Lemma 2:

check that if $X$ is any matrix, can define:

$$\underset{\substack{\uparrow \\ \text{matrix of dim } \binom{N}{r} \times \binom{N}{r}}}{(\Lambda^r X)_{I,J}} = \det(X_{I,J}) \quad \leftarrow \text{ this is the matrix which is composed of minors}$$

$$I, J \subseteq \{1, \dots N\}, \ |I| = |J| = r$$

Then $(\Lambda^r X^{*}) = (\Lambda^r X)^{*}$

$$\Lambda^r(XY) \underset{\substack{\uparrow \\ \text{Cauchy-Binet}}}{=} (\Lambda^r X)(\Lambda^r Y)$$

orthogonal \ unitary

$\Longrightarrow$ For herm. $X$, $X = UDU^*$. $\Lambda^r X = (\Lambda^r U)(\Lambda^r D)(\Lambda^r U^*)$

diagonal

gives us access to the minors.