W10D1
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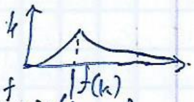
# Prob. Methods in Comb.

**Azuma's ineq.**: Let $X_0, \dots, X_m$ be a martingale with $|X_{i-1} - X_i| \le 1$ for each $i < m$. Then for every $\lambda > 0$, $\mathbb{P}(X_m > \mathbb{E}X_m + \lambda\sqrt{m}) \le e^{-\lambda^2/2}$

## Chromatic Number of $G(n, 1/2)$

We know $\chi(G) \ge \frac{|V(G)|}{\alpha(G)}$ so we'd first like to compute $\alpha(G)$ for $G \sim G(n, 1/2)$. Let $f(k) = \mathbb{E}[\#\text{copies of } K_k \text{ in } G]$ (equivalent to trying to find an empty subgraph).

Then $f(k) = \binom{n}{k} 2^{-\binom{k}{2}}$ ~~tha~~ so $\frac{f(k+1)}{f(k)} = \frac{n-k}{k+1} 2^{-k}$. Roughly 

Let $k_0 = \min(k \ge 1 \mid f(k) < 1)$. Claim – a. a. s $\omega(G) \in \{k_0 - 1, k_0, k_0 + 1\}$ $o(\log n)$

1) $k_0 \le (2+o(1))\log_2 n$ since $f(k) \le \left(\frac{en}{k}\right)^k 2^{-\frac{k-1}{2}k} << 1$ for $2^{\frac{k-1}{2}} > \frac{en}{k}$, which happens roughly at $2\log_2 n$.

2) $k_0 \ge (2-o(1))\log_2 n$ since $f(k) \ge \left(\frac{n-k}{k}\right)^k 2^{-\frac{k-1}{2}k} >> 1$ for $\frac{n-k}{k} > 2^{\frac{k-1}{2}} \Leftarrow k = (2-\varepsilon)\log_2 n$ (for large enough $n$ – $\forall \varepsilon > 0\ \exists n_0\ \forall n > n_0 \dots$).

So, $\frac{f(k_0+1)}{f(k_0)} \simeq n^{-1+o(1)}$, so $f(k_0+1) \le n^{-1+o(1)}$ and by the first moment method $G$ doesn't contain $K_{k_0+1}$ a. a. s., so $\omega(G) \le k_0$ a. a. s. (equivalently $\alpha(G) \le k$). We conclude $\chi(G) \ge (1+o(1)) \frac{n}{2\log_2 n}$ a.a.s. We prove (Bollobás) that in fact equality holds above.

**Lemma** $\mathbb{P}(\omega(G(n, 1/2)) < k_0 - 4) \le e^{-n^{2+o(1)}}$. Equality above follows: $m = \left\lfloor \frac{n}{\log^2 n} \right\rfloor$. Suffices to prove a.a.s. $\forall S, |S| = m \Rightarrow \alpha(G[S]) \ge (2-o(1))\log_2 n$. by lemma & union bound $G \sim G(m, 1/2)$ so by the lemma $\mathbb{P}(\alpha(G[S]) < k_0(m) - 4) \le e^{-m^{2+o(1)}} = e^{-n^{2+o(1)}}$, but $\#$ sets $S \le 2^n$, so the prob. the above holds for any $S$ (by union bound) $\ge 1 - 2^n e^{-n^{2+o(1)}} \ge 1 + o(1)$.

**pf. of Lemma** $Y$ = max. size of a family of edge-disjoint $K_1$ cliques ($k_1 = k_0 - 4$) Claim $\mathbb{E}(Y) \ge (1+o(1))\frac{n^2}{\dots}$

pf. claim => lemma Let $Y_0, \dots, Y_{\binom{n}{2}}$ be the edge exposure martingale, Y is Lipschitz so by Azuma

$$\mathbb{P}(\omega(G) < k_1) = \mathbb{P}(Y_{\binom{n}{2}} = 0 = \mathbb{E}Y - \mathbb{E}Y) \le e^{-\frac{(\mathbb{E}Y)^2}{2\binom{n}{2}}} = e^{-\frac{n^2}{\log^8 n}}$$

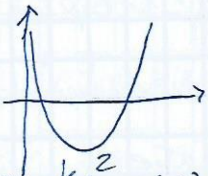
Now, $f(k_1) = f(k_0-5) = f(k_0-1)\prod_{i=1}^{4}\frac{f(k_0-i-1)}{f(k_0-i)} = f(k_0-1)\cdot n^{4+o(1)} \ge n^{4+o(1)}$.

Define W = # intersecting pairs of cliques.

Fix $0<q<1$ and take $C$ a collection of cliques obtained by selecting each $K_{k_1}$ in $G$ with prob. $q$. $C'$ is given by deleting a clique from each intersecting pair in $C$.

$\mathbb{E}(|C'|) \ge \mathbb{E}[|C|] - q^2\,\mathbb{E}[W] = q\left(f(k_1) - q\frac{\mathbb{E}[W]}{2}\right)$. Now,

$\mathbb{E}[W] = f(k_1)\underbrace{\sum_{s=2}^{k-1}\binom{k_1}{s}\binom{n-k_1}{k_1-s}2^{-\binom{k}{2}+\binom{s}{2}}}_{g(s)}$. Then $g(0) = (1+o(1))f(k_1)$ and

$\frac{g(s+1)}{g(s)} = \frac{k_1-s}{s+1}\cdot\frac{k_1-s}{n-2k_1+s+1}2^{s} \to$  So this is

$f(k_1)(1+o(1))(g(2)+g(k_1-1))$. $g(2) = \frac{k_1^2}{n}g(1) = \frac{k_1^4}{n^2}g(0) = f(k_1)\frac{k_1^4}{n^2}(1+o(1))$,

and $g(k_1-1) \le k_1 n 2^{-k_1+1} \le \frac{k_1}{n} << \frac{k_1^4}{n^2}f(k_1)$. So,

$\mathbb{E}[W] = (1+o(1))\frac{k_1^4}{n^2}f(k_1)^2$. Pick $q = \frac{n^2}{k_1^4}\frac{1}{f(k_1)} < 1$, so we get

$\mathbb{E}[|C'|] = \frac{n^2}{k_1^4}\frac{1}{f(k_1)}\left(f(k_1) - \frac{n^2}{k_1^4}\frac{1}{f(k_1)2}\frac{k_1^4}{n^2}f^2(k_1)\right) = \frac{n^2}{2k_1^4}$ as promised.

Thm. Let $p = n^{-\alpha}$, $\alpha > 5/6$. Then $\exists u = u(n,p)$ s.t. a.a.s.

$u \le \chi(G(n,p)) \le u+3$.

Lemma Fix $\overset{5/6}{\hat{\alpha}}, \hat{c}$, $p = n^{-\alpha}$. Then a.a.s every subgraph of $G(n,p)$ with $\le c\sqrt{n}$ vxs. is 3-colorable.

pf. Enough that $\forall |S| \le c\sqrt{n} \Rightarrow e_G(S) < \frac{3|S|}{2}$. $\mathbb{P}$(this won't hold for some $S$) $\le \underbrace{\sum_{s=4}^{c\sqrt{n}}\binom{n}{s}\binom{\binom{s}{2}}{3s/2}p^{\frac{3s}{2}}}_{b(s)}$. $b(s) \le \left(\frac{en}{s}\right)^s\left(\frac{es^2p}{3s}\right)^{\frac{3s}{2}} = \left(\frac{e^{5/2}ns^{1/2}p^{3/2}}{3^{3/2}}\right)^s \le$

$(cn^{-\delta})^s$ for some $\delta = \delta(\alpha)$ (since $\frac{5}{4} - \frac{3}{2}\cdot\frac{5}{6} = 0$).

pf of thm. Let $\varepsilon > 0$ be arbitrarily small and let $u$ be $\overbrace{\text{smallest}}$ such that

$\mathbb{P}(\chi(G) \le u) > \varepsilon$. $Y = \min\{|S| \mid G - S \text{ is } u\text{-colorable}\}$.

Let $Y_0, \dots Y_n$ be vxs. exposure martingale. It's vertex-Lipschitz so by Azuma, $\mathbb{P}(Y - \mathbb{E}Y \le -\lambda\sqrt{n}) \le e^{-\lambda^2/2}$. Fix $\lambda$ such that $e^{-\lambda^2/2} < \varepsilon$. Then $\mathbb{P}(Y \le \mathbb{E}Y - \lambda\sqrt{n}) < \varepsilon \le \mathbb{P}(Y(G) \le u)$, so

$\mathbb{E}Y < \lambda_0\sqrt{n}$. Claim with prob. $\geq 1-3\varepsilon$, $\chi(G) \in \{u, u+1, u+2, u+3\}$.
$\mathbb{P}(\chi(G) < u) < \varepsilon$, $\mathbb{P}(Y \geq \underbrace{2\lambda_0}_{c}\sqrt{n}) < \varepsilon$ so by lemma,
$\mathbb{P}$(Some set isn't 3-col.) $< \varepsilon$, so we can color the rest using $\leq 3$ cols. - the total probability that anything fails $\leq 3\varepsilon$.

Ex. $B$ arbitrary normed space, $v_1, \dots, v_n \in B$ $\forall i$ $\|v_i\| \leq 1$. Let $\varepsilon_1, \dots, \varepsilon_n$ i.i.d. $\mathbb{P}(\varepsilon_i = \pm 1) = 1/2$. $X = \|\sum_{i=1}^{n} \varepsilon_i v_i\|$. $\mathbb{P}(X - \mathbb{E}X \geq \lambda\sqrt{n}) <$ $[< -\lambda\sqrt{n}]$
$e^{-\lambda^2/2}$.

pf. Note that (by tri. ineq.) changing $\varepsilon_i \to -\varepsilon_i$ changes $X$ by at most 2. Therefore define $X_i = \mathbb{E}[X \mid \varepsilon_1, \dots \varepsilon_i]$ so that $X_0, \dots, X_n$ form a martingale. Therefore $|X_{i+1} - X_i| \leq 2$. But for the bound we need 1-Lipschitz! We may get a better estimate on $|X_{i+1} - X_i|$: Fix $\varepsilon \in \{-1, 1\}^n$ and let $\varepsilon'$ be $\varepsilon$ with $\varepsilon_{i+1}$ negated. Then $X_i(\varepsilon) = \frac{1}{2}(X_{i+1}(\varepsilon) + X_{i+1}(\varepsilon'))$, so $|X_i(\varepsilon) - X_{i+1}(\varepsilon)| = |X_{i+1}(\varepsilon) - X_{i+1}(\varepsilon')|/2 \leq 1$.

Indep. Bounded Differences ineq. (no proof, McDiarmid's "Concentration") Let $x$ be a vector of indep. r.v.. Suppose each $x_i \in A_i$ and $f: \prod_{i=1}^{n} A_i \to \mathbb{R}$, such that $|f(x) - f(x')| \leq c_i$ if $x, x'$ differ only on the $i^{th}$ coordinate. Then, letting $\mu = \mathbb{E}[f(x)]$, for every $t \geq 0$, $\mathbb{P}(f(x) - \mu \geq t) \leq e^{\frac{-2t^2}{\sum_{i=1}^{n} c_i^2}}$.

Application: Isoperimetric ineq. of the hypercube
$\Omega = \Omega_1 \times \dots \times \Omega_n$. Given $x, y \in \Omega$ the Hamming dist.
$d_H(x,y) = \#\{i \mid x_i \neq y_i\}$.

Thm. Let $(x_1, \dots, x_n)$ be ind. r.v., $x_i \in \Omega_i$, $A \subseteq \Omega$ arbitrary. Then for any $t \geq 0$, $\mathbb{P}(x \in A)\underbrace{\mathbb{P}(d_H(x,A) \geq t)}_{A_t} \leq e^{-t^2/2n}$.

Cor. $A \subseteq \{0,1\}^n$, $|A| \geq \varepsilon \cdot 2^n$, $\frac{|A_t|}{2^n} \geq 1 - \frac{e^{-t^2/2n}}{\varepsilon}$.

pf. Let $\rho = \mathbb{P}(x \in A)$, $\mu = \mathbb{E}[d_H(x,A)]$. By IBDI for $t \geq 0$,
$\mathbb{P}(d_H(x,A) - \mu \geq t) \leq e^{-2t^2/n}$. Note $\rho = \mathbb{P}(d_H(x,A) - \mu \overset{\leq}{=} -\mu) \leq e^{-2\mu^2/n}$, so $\mu \leq \sqrt{\frac{1}{2} n \log \frac{1}{\rho}} = t_0$. Then

$\mathbb{P}(d_H(x,A) \geq t_0 + t) \leq e^{-2t^2/n}$. For $t \geq t_0$, $\mathbb{P}(d_H(x,A) \geq t) \leq e^{-2(t-t_0)^2/n}$.
If $t \geq 2t_0$, $(t-t_0)^2 \geq t^2/4$ so $\mathbb{P}(d_H(x,A) \geq t) \leq e^{-t^2/2n}$. If $t \leq 2t_0$
recall $\rho = \mathbb{P}(x \in A) = e^{-2t_0^2/n}$. Therefore $\min\{\mathbb{P}(x \in A), \mathbb{P}(d_H(x,A) \geq t)\}$
is $\leq e^{t^2/2n}$.

Now, for any $\alpha = (\alpha_1, \dots, \alpha_n)$ we may define $d_\alpha(x,y) = \sum_{x_i \neq y_i} \alpha_i$
($d_H$ corresponds to $\forall i\ \alpha_i = 1$). Using the same method we may obtain $\mathbb{P}(x \in A)\mathbb{P}(d_\alpha(x,A) \geq t) \leq e^{\frac{-t^2}{2\sum \alpha_i^2}}$.

<u>Talagrand's ineq.</u> (no pf.) $\Omega, x, A$ like before. Define $\rho(x,A) = \sup_{\|\alpha\|=1} \min_{y \in A} d_\alpha(x,y)$. In particular, $\rho(x,A) \geq d_\alpha(x,A)$. Then $\forall A, \forall t \geq 0$, $\mathbb{P}(x \in A)\mathbb{P}(\rho(x,A) \geq t) \leq e^{-t^2/4}$.

<u>Lemma (Talagrand)</u> $\int_\Omega \exp\left(\frac{1}{4}\rho^2(x,A)\right)dx \leq \frac{1}{\mathbb{P}(x \in A)}$. pf by induction on $n$ using an alternative definition of $\rho$ as a distance from the convex hull of a set in the product space. Given the lemma, the bound is just Chernoff bounding.