~~Beda~~ W8D1 14/12/14

Prop. Methods in Comb.

Recall: X dist. Lattice, $\mu: X \to \mathbb{R}_{\geq 0}$ is log supermodular.

FKG ineq. For increasing $f, g: X \to \mathbb{R}_{\geq 0}$, $\left(\sum_{x\in X} f(x)\mu(x)\right)\left(\sum_{x\in X} g(x)\mu(x)\right) \leq \left(\sum_{x\in X} f(x)g(x)\mu(x)\right)\left(\sum_{x\in X}\mu(x)\right)$.

Thm (XYZ) P ~~poset~~ finite poset, $a_1, a_2, a_3 \in P$. Let Q be a linear extention of ~~P~~ P chosen UAR.

Then $\mathbb{P}(a_1 \leq a_3 \ \& \ a_2 \leq a_3) \geq \mathbb{P}(a_1 \leq a_2)\mathbb{P}(a_1 \leq a_3)$.

pf. Assume $P = \{a_1, \dots a_n\}$. Fix $m >> n$ which will be specified later, $L = \{1, \dots, m\}^n$. Define $\leq$ on L:

$(x_1, \dots, x_n) \leq (y_1, \dots, y_n) \iff x_1 \geq y_1 \ \& \ \forall i \geq 2 \ x_i - x_1 \leq y_i - y_1$.

claim $(L, \leq)$ forms a lattice.

pf. $(x_1, \dots, x_n) \vee (y_1, \dots, y_n) = (\min\{x_1, y_1\}, \min\{x_1, y_1\} + \max\{x_2 - x_1, y_2 - y_1\}, \dots)$,

$(x_1, \dots, x_n) \wedge (y_1, \dots, y_n) = (\max\{x_1, y_1\}, \max\{x_1, y_1\} + \min\{x_2 - x_1, y_2 - y_1\}, \dots)$

satisfy the conditions of join & meet.

$(L, \leq)$ is also distributive.

Now, define μ the "char. function of P": $\mu(x_1, \dots x_n) = 1 \iff \forall i,j$ s.t. $a_i \leq_P a_j$, $x_i \leq x_j$. μ is log-supermodular since if x, y preserve the order on P, so do $x \vee y$ and $x \wedge y$.

Now, every lin. extention $Q \supseteq P$, if $Q_{i_1} \leq Q_{i_2} \dots \leq Q_{i_n}$, then it is "represented" in L by some fixed num. of elements $\bar{x}$ s.t. $\mu(\bar{x}) = 1$. Such a rep. may not uniquely determine Q, as equality may occour (but with very small prob. for large enough m). However, $\bar{x} \in \mu^{-1}(1)$ which doesn't contain repeated elements uniquely determines such Q, each Q being represented by exactly such $\binom{m}{n}$ $\bar{x}$ values. Therefore

$\frac{1}{\binom{m}{n}} \mu|_{\mu^{-1}(1) \cap \{\text{no repeated elements}\}}$ is the uniform measure over linear extensions Q of P.

Define f, g, $f(\bar{x}) = \begin{cases} 1 & x_1 \leq x_2 \\ 0 & o/w \end{cases}$ $g(\bar{x}) = \begin{cases} 1 & x_1 \leq x_3 \\ 0 & o/w \end{cases}$ By FKG

(non increasing)

$\mathbb{P}_L(x_1 \leq x_2)\mathbb{P}_L(x_1 \leq x_3) \leq \mathbb{P}_L(x_1 \leq x_2 \& x_1 \leq x_3)$. But $\mathbb{P}_L \neq$ uniform measure on lin. extensions. However, $\mathbb{P}(\exists i,j \text{ s.t. } x_i = x_j) \leq$

$\frac{\binom{n}{2} m \cdot m^{n-2}}{\frac{1}{n!} m^n} \leq \frac{n! \binom{n}{2}}{m} \searrow_{m\to\infty} 0$. Therefore $\mathbb{P}_L(A) \to \mathbb{P}_m(A)$ for the uniform measure, which gives the result.

Concentration of Measure Inequalities

Z a r.v. with $\mathbb{E}Z$ exists. We want to estimate $\mathbb{P}(Z - \mathbb{E}Z \geq t)$ or $\mathbb{P}(Z - \mathbb{E}Z \leq -t)$, for some $t > 0$.

Markov: $\mathbb{P}(Y \geq t) \leq \frac{\mathbb{E}Y}{t}$ (can let $Y = |Z - \mathbb{E}Z|$) for $t > 0$, $Y \geq 0$.

Choose $\varphi: \underset{\text{range}(Y)}{\underset{\cup}{I}} \to \mathbb{R}_{\geq 0}$ increasing. Then $\mathbb{P}(Y \geq t) =$

$= \mathbb{P}(\varphi(Y) \geq \varphi(t)) \leq \frac{\mathbb{E}\varphi(Y)}{\varphi(t)}$.

A common choice, $\varphi(t) = t^2$, gives Chebyshev's inequality.
We can generalize $\varphi(t) = t^q$ $(q > 0)$ which gives the moment method - but $\mathbb{E}\varphi(Y)$ may be difficult to calculate.
If $Z = X_1 + \dots + X_n$ for X_i, X_j indep., we can take $\varphi(t) = e^{\lambda t}$.
Using Chebyshev we would get (assuming X_i i.i.d. with $\mathrm{Var}\, X_i = \sigma^2$)
$\mathbb{P}(|Z - \mathbb{E}Z| \geq t) \leq \frac{n\sigma^2}{t^2}$, which goes to 0 quad. in t, very far from the truth (exp. in t^2...).
Choose $\lambda > 0$, $\varphi(t) = e^{\lambda t}$. Then $\mathbb{P}(Z - \mathbb{E}Z \geq t) \leq \frac{\mathbb{E}(\varphi(Z - \mathbb{E}Z))}{\varphi(t)}$.
$\mathbb{E}(\varphi(Z - \mathbb{E}Z)) = \frac{\prod_{i=1}^n \mathbb{E}e^{\lambda X_i}}{\prod_{i=1}^n e^{\lambda \mathbb{E}X_i}}$. So this gives a bound
$\frac{\prod_{i=1}^n \mathbb{E}e^{\lambda X_i}}{e^{\lambda t + \lambda \sum \mathbb{E}X_i}}$ The Cramér - Chernoff method.

Thm. Suppose $X_1, \dots, X_n$ i.i.d. uniform on $\{\pm 1\}$.
Let $S_n = X_1 + \dots + X_n$. Then, for any $a > 0$,
$\mathbb{P}(S_n > a) = \mathbb{P}(-S_n < a) \leq e^{\frac{-a^2}{2n}}$. This is only good for at least $a \approx \sqrt{n} = \sqrt{\mathrm{Var}(S_n)}$, the same range of Chebyshev, but gives much better results in this range.

pf. $\mathbb{E}e^{\lambda X_i} = \frac{e^{\lambda} + e^{-\lambda}}{2} \leq e^{\lambda^2/2}$ by comparing the Taylor series.
This gives a bound of $\frac{e^{\frac{\lambda^2 n}{2}}}{e^{\lambda a}} = e^{\lambda(\frac{\lambda n}{2} - a)}$. Take $\lambda = \frac{a}{n}$ to get

$e^{\frac{-a^2}{2n}}$ as promised. In fact $\frac{S_n}{\sqrt{n}} \xrightarrow{N} N(0,1)$ so

$\mathbb{P}(S_n > a_n) = \mathbb{P}\left(\frac{S_n}{\sqrt{n}} > \frac{a_n}{\sqrt{n}}\right) \xrightarrow[n\to\infty]{} \mathbb{P}(N(0,1) > a) \approx \frac{e^{-a^2/2}}{\sqrt{2\pi}a}$ – ($a = \lim \frac{a_n}{\sqrt{n}}$)

very close to the Chernoff bound.

Non-Symm. case

$p_1, \ldots, p_n$, so that $\mathbb{P}(X_i = 1-p_i) = p_i$, $\mathbb{P}(X_i = -p_i) = 1-p_i$ (so that $\mathbb{E}X_i = 0$). Let $p = \frac{\sum_{i=1}^n p_i}{n}$, $a>0$. Then for $X = \sum_{i=1}^n X_i$,

$\mathbb{P}(X > a) \leq \exp\left(\frac{-a^2}{2pn} + \frac{a^3}{2(pn)^2}\right)$, $\mathbb{P}(X < -a) \leq \exp\left(\frac{-a^2}{2pn}\right)$.

Cor ~~Under the same assumptions~~, let Y be a sum of ind. (both independent & indicator) r.v. $\forall \varepsilon > 0 \ \exists c_\varepsilon > 0$ (indep. of Y) s.t. $\mathbb{P}(|Y - \mathbb{E}Y| \geq \varepsilon \mathbb{E}Y) \leq 2e^{-c_\varepsilon \mathbb{E}Y}$.

Martingales

A sequence $X_0, \ldots X_m$ is a *martingale* if each X_i is integrable and $\mathbb{E}[X_i | X_0, \ldots, X_{i-1}] = X_{i-1}$.

Examples of Martingales:

(1) $X_i = \sum_{k=1}^{i} Y_k$, where $Y_1, \ldots, Y_m$ are indep. with $\mathbb{E}Y_i = 0$.

(2) Doob Martingale – X r.v. $\mathcal{F}_0 \subseteq \mathcal{F}_1 \ldots \subseteq \mathcal{F}_n$, $X_i = \mathbb{E}[X | \mathcal{F}_i]$. ($\mathcal{F}_0 = \{\emptyset, \Omega\}$)

(3) Edge exposure martingale [$G \sim G(n,p)$, $f: \text{Graphs} \to \mathbb{R}$] – order the $m = \binom{n}{2}$ pairs of vxs. $e_1, \ldots, e_m$. Define $\mathbb{E}[f | G \cap \{e_1, \ldots, e_i\}] = X_i$.

(4) Vertex exposure martingale – order the vertices $v_1, \ldots, v_n$. Define $X_i = \mathbb{E}(f | G[\{v_1, \ldots, v_i\}])$.

Azuma's inequality

Let $0 = X_0, X_1, \ldots, X_m$ be a martingale s.t. $|X_{i+1} - X_i| \leq 1$ a.s. Then $\mathbb{P}(X_m > \lambda\sqrt{m}) \leq e^{-\lambda^2/2}$.

Def. A graph func. is said to be:

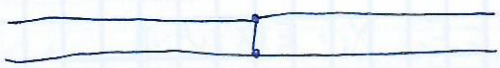
(a) edge-Lipshitz $\iff |f(H) - f(H \cup \{e\})| \leq 1$.

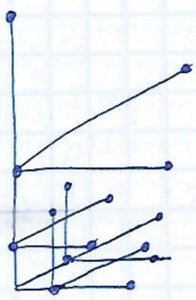
(b) vertex-Lipschitz $\iff |f(H) - f(H \cup \{e_1, \ldots, e_n\})| \leq 1$ whenever $v \in e_1, \ldots e_n$ for some v.

Fact When f satisfies vx/edge Lipschitz condition, the vx/edge exposure martingale satisfies $|X_{i+1} - X_i| \leq 1$.

Cor. (Shamir & Spencer) For arbitrary n, p, $G \sim \mathcal{G}(n,p)$, then $c = \mathbb{E}[\chi(G)]$. Then $\mathbb{P}[|\chi(G) - c| \geq \lambda\sqrt{n-1}] \leq 2e^{-\lambda^2/2}$.

Azuma pf. (Sketch) Let $Y_i = X_i - X_{i-1}$. $|Y_i| \leq 1$ a.s. and $\mathbb{E}\, Y_i = 0$.
$\mathbb{E}(e^{\alpha Y_i} \mid X_1, \dots, X_{i-1}) \leq e^{\alpha^2/2}$, and we can just follow the pf. for Bernoulli r.v.





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