W7D1
7/12/14

Prob. Methods in Comb.

[dependent random choice]

Lemma - if $\varepsilon>0$, $d\le n$ are integers, G graph with $N>4d\varepsilon^{-d}n$ vertices and at least $\varepsilon N^2/2$ edges, then $\exists U\subseteq V(G)$, $|U|>2n$, s.t. $|\{S\in\binom{U}{d} : |N^*(S)|<n\}|<(2d)^{-d}\binom{|U|}{d}$.

Aim: $r(Q_d)\le d\cdot 2^{d+3}$.

Thm. Let H be bipartite with n vertices and $\Delta(H)\le\Delta$. If $\varepsilon>0$, G is a graph with $N\ge 8\Delta\varepsilon^{-\Delta}$ vertices and at least $\varepsilon\binom{N}{2}$ edges, then $H\subseteq G$.

Lemma H bipartite, n vxs, $\Delta(H)\le d$. If $\exists U\subseteq V(G)$ $\overbrace{|U|>2n}$ s.t. $|\underbrace{\{S\in\binom{U}{d} \mid |N^*(S)|<n\}}_{B}|\le(2d)^{-d}\binom{|U|}{d}$ then $H\subseteq G$.

pf. Can't naively embed like before.

Call $T\subseteq U$ with $|T|\le d$ good if $\{S\in B \mid T\subseteq S\}=B_T$ satisfies $|B_T|\le(2d)^{-d}\binom{|U|}{d-|T|}$. For a good T, let X_T be $\{x\in U \mid \{x\}\cup T$ isn't good$\}$. If $|T|=d$, T good $\Leftrightarrow$ $N^*(T)\ge n$.
$|T|<d$

Goal embed V_1 into U in such a way that the images of $N(w)$ $w\in V_2$, are all good sets.

$V=\{v_1,\dots,v_k\}$, $L(i)=\{v_1,\dots,v_i\}$. Use induction on I:

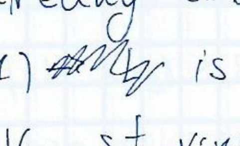
For each i, can embed $L(i)$ into U s.t. $\forall u\in V_2$, $N(w)\cap L_i$ is good.

pf $i=0$: $L(0)=\phi$ just need to prove ϕ is good. This is true by def.

step $i\to i+1$: suppose $L(i)$ is already embedded v_{i+1}.

If $v_i\notin N(w)\to N(w)\cap L(i)=N(w)\cap L(i+1)$ is good

There are at most d vxs $w\in V_2$ s.t. $v_{i+1}\in N(w)$, call them $w_1,\dots,w_k$. $W_j=L(i)\cap N(w_j)$ (good).



$N(w_j)\cap L(i+1)=W_j\cup\{v_{i+1}\}$.

Enough to find $u\in U\setminus\varphi(L(i))$ s.t. $W_j\cup\{u\}$ is good for $i=1,\dots,d$. That is show $U\setminus[(X_{w_1},\dots,X_{w_k})\cup(\varphi(L(i)))]\ne\phi$.

Enough to show $|U|-|X_{w_1}|\ \dots\ -|X_{w_k}|-|L(i)|$, $|T|\le d-1$ is good, __Claim__ $|X_T| \leq \frac{|U|}{2d}$. $|X_T| \cdot \frac{(2d)^{|T|+1-d}\binom{U}{d-|T|-1}}{d-|T|} \leq |B_T| \leq (2d)^{|T|-d}\binom{|U|}{d-|T|}$.

so $|X_T| \leq \frac{1}{2d}\frac{(d-|T|)\binom{|U|}{d-|T|}}{\binom{|U|}{d-|T|-1}} = |U| - (d-|T|-1) \leq |U|$

We constructed an embedding $\varphi: V_1 \to U$ s.t. $\forall w \in V_2$, $\varphi(N(w))$ is good. In particular $|N^*(\varphi(N(w)))| \geq n$

so we can just embed V_2 one by one:

$a \in V_2$, $w \in N^*(\underbrace{\varphi(N(w))}_{\geq n}) \setminus (\varphi(V_1) \cup \underbrace{\varphi(\text{Prev. embedded } V_2)}_{\leq n-1}) \geq 1$.

$\mathcal{H}$ = hamiltonian, $\mathcal{P}$ = planar, $G \sim G(n, 1/2)$.

$\mathbb{P}(G \in \mathcal{P} \cap \mathcal{H}) \overset{?}{} \mathbb{P}(G \in \mathcal{P})\,\mathbb{P}(G \in \mathcal{H})$. This seems intuitive $\boxed{\leq}$ (Kleitman's lemma) but the proof isn't immediate.

__Thm.__ $\mathcal{P}(\underbrace{\{1, \dots, n\}}_{[n]})$. $\varphi: \mathcal{P}([n]) \to \mathbb{R}$. If $\mathcal{A} \subseteq \mathcal{P}([n])$, $\varphi(\mathcal{A}) = \sum_{A \in \mathcal{A}} \varphi(A)$. $\mathcal{A} \cup \mathcal{B} = \{A \cup B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$ and sim. for intersection. Let $\alpha, \beta, \gamma, \delta: \mathcal{P}([n]) \to \mathbb{R}_{\geq 0}$. Suppose $\forall A, B \subseteq [n]$ $\alpha(A)\beta(B) \leq \gamma(A \cup B)\delta(A \cap B)$. Then $\forall \mathcal{A}, \mathcal{B} \subseteq \mathcal{P}([n])$ $\alpha(\mathcal{A})\beta(\mathcal{B}) \leq \gamma(\mathcal{A} \cup \mathcal{B})\delta(\mathcal{A} \cap \mathcal{B})$.

__pf.__ induction on n. WLOG we may assume $\mathcal{A} = \mathcal{B} = \mathcal{P}([n])$ (Just put $\alpha(A) = 0$ for $A \notin \mathcal{A}$, $\beta(B) = 0$ for $B \notin \mathcal{B}$, ...).

Base $n=1$, let $\varphi_0 = \varphi(\emptyset)$, $\varphi_1 = \varphi(\{1\})$. We need

(*) $(\alpha_0 + \alpha_1)(\beta_0 + \beta_1) \leq (\gamma_0 + \gamma_1)(\delta_0 + \delta_1)$, given

$\alpha_0\beta_0 \leq \gamma_0\delta_0$, $\alpha_0\beta_1 \leq \gamma_1\delta_0$, $\alpha_1\beta_0 \leq \gamma_1\delta_0$, $\alpha_1\beta_1 \leq \gamma_1\delta_1$

If $\gamma_1 = 0$ or $\delta_0 = 0$ we are done. O/W, (*) ~~is eq. to~~ is implied by:

$(\alpha_0 + \alpha_1)(\beta_0 + \beta_1) \leq \left(\frac{\alpha_0\beta_0}{\delta_0} + \gamma_1\right)\left(\delta_0 + \frac{\alpha_1\beta_1}{\gamma_1}\right)$ $/ \cdot \delta_0\gamma_1$

$\delta_0\gamma_1(\alpha_0 + \alpha_1)(\beta_0 + \beta_1) \leq (\alpha_0\beta_0 + \gamma_1\delta_0)(\delta_0\gamma_1 + \alpha_1\beta_1)$

$\Updownarrow$

$(\gamma_1\delta_0 - \alpha_0\beta_1)(\gamma_1\delta_0 - \alpha_1\beta_0) \geq 0$ which holds.

Step: $n \to n+1$. $\varphi \in \{\alpha, \beta, \gamma, \delta\}$ Defined on $\mathcal{P}([n+1])$. Define

$\varphi': \mathcal{P}([n]) \to \mathbb{R}$ by $\varphi'(A) = \varphi(A) + \varphi(A \cup \{n+1\})$. Note $\varphi'(\mathcal{P}[n]) = \varphi(\mathcal{P}(n+1))$. Just need to check that the assumptions hold for $\alpha', \beta', \gamma', \delta'$.

Take $A, B \subseteq [n]$. We need $\alpha'(A)\beta'(B) \leq \gamma'(A \cup B)\delta'(A \cap B)$.

But that is just $(\underset{\alpha_0}{\alpha(A)} + \underset{\alpha_1}{\alpha(A \cup \{n+1\})})(\underset{\beta_0}{\beta(B)} + \underset{\beta_1}{\beta(B \cup \{n+1\})}) \leq$
$(\underset{\gamma_0}{\gamma(A \cup B)} + \underset{\gamma_1}{\gamma(A \cup B \cup \{n+1\})})(\underset{\delta_0}{\delta(A \cap B)} + \underset{\delta_1}{\delta(A \cap B \cup \{n+1\})})$ and this is identical to the base case.

A <u>lattice</u> is an ordered set $(X, \leq)$ such that every $x, y \in X$ has a unique smallest element greater than both, $x \vee y$ ("join") and greatest element smaller than both $x \wedge y$ ("meet").

A lattice is <u>distributive</u> iff $\forall x, y, z \in X$
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

If $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ closed to union & intersection is a distributive lattice WRT inclusion (with $\vee = \cup$, $\wedge = \cap$). In fact, $\forall$ such finite lattices are of this kind.

<u>Cor.</u> X finite dist. lattices with $\alpha, \beta, \gamma, \delta : X \to \mathbb{R}_{\geq 0}$ s.t.
$\forall x, y \in X$, $\alpha(x)\beta(y) \leq \gamma(x \vee y)\delta(x \wedge y)$ then $\forall A, B \subseteq X$
$\alpha(A)\beta(B) \leq \gamma(A \vee B)\delta(A \wedge B)$.

<u>Cor</u> X finite dist. lattice with $A, B \in X$, then
$|A|\,|B| \leq |A \vee B|\,|A \wedge B|$

<u>Cor</u> For $\mathcal{A}$ a finite family of sets, $|\mathcal{A} \setminus \mathcal{A}| \geq |\mathcal{A}|$.

<u>pf.</u> Just take $A = \mathcal{A}$, $B \in \mathcal{A}$.

<u>The FKG inequality</u>

X: fin. dist. lattice. Call a func. $\mu : X \to \mathbb{R}_{\geq 0}$ <u>log-supermodular</u> iff $\forall x, y \in X$, $\mu(x)\mu(y) \leq \mu(x \vee y)\mu(x \wedge y)$.

$f : X \to \mathbb{R}_{\geq 0}$ is <u>increasing</u> iff $f(x) \leq f(y)$ $\forall x, y \in X$ $x \leq_X y$.

<u>Thm.</u> X fin. dist. lattice, μ log-supermodular, f, g increasing then $\left[\sum_{x \in X} f(x)\mu(x)\right]\left[\sum_{x \in X} g(x)\mu(x)\right] \leq \left[\sum_{x \in X} f(x)g(x)\mu(x)\right]\left[\sum_{x \in X} \mu(x)\right]$.

WLOG one can assume $\sum_{x\in X}\mu(x)=1$. Then, when thinking of μ as a prob. measure, $E_\mu[f]\,E_\mu[g]\leq E_\mu[fg]$.

pf. Let $\alpha(x)=f(x)\mu(x)$, $\beta(x)=g(x)\mu(x)$, $\gamma(x)=f(x)g(x)\mu(x)$, $\delta(x)=\mu(x)$,
we just need
then $\alpha(x)\beta(y)=f(x)\mu(x)g(y)\mu(y)\overset{?}{\leq} f(x\vee y)g(x\vee y)\mu(x\vee y)\mu(x\wedge y)$ $= \gamma(x\vee y)\delta(x\wedge y)$.

But $\mu(x)\mu(y)\leq\mu(x\vee y)\mu(x\wedge y)$, $f(x)\leq f(x\vee y)$ and $g(y)\leq g(x\vee y)$. Multiply those 3 inequalities to get the above. [Same holds if f,g are decreasing taking $\gamma(x)=\mu(x)$, $\delta(x)=f(x)g(x)\mu(x)$...]

Cor (Kleitman's Lemma) Let $\mathcal{A},\mathcal{B}\subseteq\mathcal{P}([n])$ be increasing in $\mathcal{C},\mathcal{D}\subseteq\mathcal{P}([n])$ decreasing. Then

(1) $\frac{|\mathcal{A}\cap\mathcal{B}|}{2^n}\geq\frac{|\mathcal{A}|}{2^n}\frac{|\mathcal{B}|}{2^n}$. (2) $\frac{|\mathcal{A}\cap\mathcal{C}|}{2^n}\leq\frac{|\mathcal{A}|}{2^n}\frac{|\mathcal{C}|}{2^n}$. (3) $\frac{|\mathcal{C}\cap\mathcal{D}|}{2^n}\geq\frac{|\mathcal{C}|}{2^n}\frac{|\mathcal{D}|}{2^n}$.

pf. Take $X=(\mathcal{P}([n]),\subseteq)$, $f=\mathbb{1}_{\mathcal{A}}$, $g=\mathbb{1}_{\mathcal{B}}$ [μ const.]

Given two lattices X,Y, one may define $Z=(X\times Y,\leq_X\times\leq_Y)$. This is a lattice with $(x,y)\overset{\wedge}{\vee}(x',y')=(x\overset{\wedge}{\vee}x',y\overset{\wedge}{\vee}y')$. If X,Y are dist. so is Z, and mult. of log-sup. funcs. is log-sup.

$\mu:\{0,1\}\to\mathbb{R}_{\geq0}$ given by $\mu(1)=p$, $\mu(0)=1-p$ is log-sup, so if Ω is finite, $\forall\omega\in\Omega$ $p_\omega\in[0,1]$. Define a prob. measure on $\mathcal{P}(\Omega)$ by $\mu(A)=\prod_{\omega\in A}p_\omega\cdot\prod_{\omega\notin A}(1-p_\omega)$. μ is log-sup. If particular FKG applies, giving the following version of Kleitman's Lemma:

Cor. Let $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\subseteq\mathcal{P}([n])$, $\mathcal{A},\mathcal{B}$ inc. $\mathcal{C},\mathcal{D}$ dec., $p_1,\dots,p_n\in[0,1]$. Define $\mathbb{P}(X)=\prod_{x\in X}p_x\prod_{x\notin X}(1-p_x)$. Then $\mathbb{P}(\mathcal{A}\cap\mathcal{B})\geq\mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B})$, $\mathbb{P}(\mathcal{C}\cap\mathcal{D})\geq\mathbb{P}(\mathcal{C})\mathbb{P}(\mathcal{D})$, $\mathbb{P}(\mathcal{A}\cap\mathcal{C})\leq\mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{C})$.

Let X be a (finite) POset. and let $\mathcal{L}(X)$ be the family of linear orderings of X.

Thm. (XYZ). Choose $L\in\mathcal{L}(X)$ uni. $\mathbb{P}(x\leq y\wedge x\leq z)\overset{?}{\geq}\mathbb{P}(x\leq y)\mathbb{P}(x\leq z)$.

This follows from the general theory of lattices... (no pf.).