W6 D1
30/11/14

~~Pf~~ Prob. Method in Comb.

More on Lower Bounds of Ramsey Numbers

Recent result - algorithmic ~~pf~~ proof of the local lemma in pol. time.

Let $x_1, \dots, x_m$ be r.v.s and each x_i takes values in Ω_i. $A_1, \dots, A_n$ events ($A_j \subseteq \prod_{i=1}^{n} \Omega_i$). $j \mapsto vars(A_j) = I_j \subseteq \{1, \dots, m\}$ s.t. A_j depends only on I_j.

Suppose $x_1, \dots, x_n$ are indep. We then define a dep. graph G as follows: $V(G) = \{1, \dots, n\}$, $E(G) = \{(j_1, j_2) \mid I_{j_1} \cap I_{j_2} \neq \phi\}$.

(*) $\exists x_1, \dots, x_n < 1$ s/t $\forall j \leq x_j \prod_{k \sim j} (1 - x_k)$. [LLL: (*) $\rightarrow P(\bigcap_{j=1}^{n} \bar{A}_j) \geq \prod_{j=1}^{n} (1 - x_j) > 0$.]

Algorithm RESAMPLE:

(1) Assign rand. vals. to $x_1, \dots, x_n$.

(2) While $\exists j$, A_j holds in the current evaluation, resample $\forall x_i$ s/t $i \in I_j$ for the smallest such j.

Thm: (*) $\longrightarrow$ RESAMPLE terminates in expected sum $\leq \sum_{i=1}^{n} \frac{x_i}{1 - x_i}$.
(won't prove)

Entropy Compression Method: Example

Let f be a coloring of G. A simple path of length $2r$ is repetitive if $\forall i \in \{1 \dots r\}$, $f(v_i) = f(v_{i+r})$.
f is non-repetitive if it has no repetitive path.

$\pi(G)$ = min. num of colors in a non-rep. coloring of G.

Thm $\pi(P_n) = 3$ for $n \geq 4$.

pf. define $s(a) = abcab$, $s(b) = acabcb$, $s(c) = acbcacb$.

w non-rep. $\rightarrow s(w)$ non. rep.

List-nonrep. color - $\pi_\ell(G)$. Clearly $\pi(G) \leq \pi_\ell(G)$, so $\pi_\ell(P_n) \geq 3$.

Thm. $\pi_\ell(P_n) \leq 4$.

pf. Using the following alg.:

1) Let $S=\emptyset$.

2) while $|S|<n$, let r be a random symbol from $L(|S|+1)$.

If $S\cdot r$ is non-rep, $S \leftarrow S\cdot r$.

else $\underbrace{S\cdot r = S'\circ w\circ w}_{\exists! w}$, then $S \leftarrow S'\circ w$.

Suppose it never terminates. Choose $M \gg n$ and run it for M steps. #diff. executions $= 4^M$.

But also, we can keep a "log" of how many symbols we deleted (0 or $|w|$).

After M steps we write the final word s ($|s|<n$).

This allows us to recover the entire history of the M execution steps. So $4^M = \#\text{logs}$.

$\#\text{logs} \le 4^{n-1} \cdot \#$ history charts.

If $d = \#$ symbols deleted, $\#$ history charts $\le \binom{M+d}{d} = \binom{2M-\ell}{M}$

where $\ell = M-d$. So $\#\text{logs} \le \sum_{\ell=1}^{n-1} 4^{\ell} \cdot \sum_{\ell=1}^{n-1} \binom{2M-\ell}{M} \le \sum_{\ell=1}^{n-1} 4^{\ell} \cdot n\binom{2M}{M}$.

So, $\#\text{logs} \le 4^n \cdot n \cdot \frac{4^M}{c\sqrt{M}} < 4^M$ for large enough M.

Dependent Random Choice

$\mathrm{Var}(SAT\text{-}LIT) = \frac{3}{4}$

Lemma let a, d, m, n, r be integers. Let G be a n-vx. graph with avg. deg. d. If $\exists t>0$ integer with $\frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t \ge a$, then $V(G)$ contains a set of $|U| \ge a$ vertices s/t every set of r vertices in U have at least m common neighbours.

pf. Pick t vertices indep. at random with reps. Let $U = |N(v_1) \cap N(v_2) \dots \cap N(v_t)|$. $\mathbb{E}[|U|] = \sum_{v\in V} P(v\in U) = \sum_{v\in V(G)} \left(\frac{\deg(v)}{n}\right)^t \ge$ ($x \mapsto x^t$ convex)

$n\cdot\left(\frac{d}{n}\right)^t = \frac{d^t}{n^{t-1}}$. $\mathbb{E}(\#\, r\text{-sets in } U \text{ with } <m \text{ common neighbours}) \le \binom{n}{r}\left(\frac{m}{n}\right)^t$. So we remove 1 vx. from each such set and get U.

H graph. Define $ex(n,H) = \max\{e(G) \mid v(G)=n,\ H \not\subseteq G\}$.

e.g. $ex(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor$.

Erdős – Stone: $ex(n,H) = \left(1 - \frac{1}{\chi(H)-1}\right)\binom{n}{2} + o(n^2)$. Solves the problem ~~if~~ unless H is bipartite. $\forall s \leq t,\ ex(n, K_{s,t}) \leq C n^{2-1/s}$.

Thm. Suppose H is bipartite with $\underset{|A|}{a} + \underset{|B|}{b}$ vxs. Suppose moreover $\forall b \in B$, $\deg b \leq s$. Then $ex(n,H) \leq C(H)\, n^{2-1/s}$.

pf. Suppose $e(G) \geq C \cdot n^{2-1/s}$. Suppose we find a set $a \leq |U| \subseteq V(G)$ of ~~edges~~ vxs. s/t each s-subset of U has at least $a+b$ common neighbours. Then map $A \overset{\varphi}{\hookrightarrow} U$ and B into $G \setminus \varphi(A)$ as follows:

Let $B = \{v_1, \dots, v_b\}$. For $i=1,\dots,b$ do the following:

Let $X = \varphi(N(v_i)) \subseteq U$. Then $|X| \leq s$ so $|N^*(X)| \geq a+b$.

map v_i to an arb. element of $N^*(X) \setminus \underbrace{\varphi(A) \cup \{v_1, \dots, v_{i-1}\}}_{<a+b \text{ so it's not empty.}}$

We need to find such U - but according to the lemma (with $a \leftarrow a$, $m \leftarrow a+b$, $r \leftarrow s$, $d \leftarrow C \cdot n^{1-1/s}$, $t \leftarrow s$). The condition is $\text{LHS}(*) \geq C^s - \frac{n^s}{s!}\left(\frac{a+b}{n}\right)^s \geq a$ for C sufficiently large.

Ramsey Number $R(H) = \min\{n \mid \forall$ 2-coloring of $E(K_n)$ contains a monochromatic copy of $H\}$. Well-defined since $R(H) \leq R(|V(H)|)$.

Q_r is the r-dimensional cube graph. What is $R(Q_r)$? conj $R(Q_r) \leq C \cdot 2^r$, proved but is quite complicated, for 2^{2r}.

Thm $R(Q_r) \leq 2^{3r}$. ~~Suppose~~

pf. $N = 2^{3r}$. 2-color $E(K_N)$ and let G be the denser color, with $E(G) \geq \frac{1}{2}\binom{N}{2} \geq 2^{-7/3} N^2$. We claim $Q_r \subseteq G$.

Therefore $\underset{\text{avg. degree}}{d} \geq 2^{-4/3} N$. (assume r even)

Let $t \leftarrow \frac{3}{2} r$, $m \leftarrow 2^r$, $a \leftarrow 2^{r-1}$.

$(*) - \frac{d^t}{N^{t-1}} - \binom{N}{r}\left(\frac{m}{N}\right)^t \geq N \cdot 2^{-4t/3} - \frac{N^{r-t}\, m^t}{r!} \geq 2^r - \frac{2^{3r^2 - 2rt}}{r!} \geq 2^r - 1$.

By lemma, $\exists U \subseteq V(G)$, $|U| \geq 2^{r-1}$ s/t every r-set of U has $\geq 2^r$ common neighbours. Q_r is bipartite with 2^{r-1} vertices in each side A, B. Embedd $A \overset{\varphi}{\hookrightarrow} U$ and $B \to G \setminus \varphi(U)$ like before, so G has a copy of Q_r.

Our Goal $R(Q_r) \leq r 2^{2r+3} \approx 8 \cdot (2^r)^2 \cdot \log(2^r)$.

Thm. Let H be a bipartite graph with n vxs, $\Delta(H) = \Delta \geq 1$. If $\varepsilon > 0$ and G is a graph with $N \geq 8\Delta \varepsilon^{-\Delta} n$ vxs and at least $\varepsilon \binom{N}{2}$ edges, $H \subseteq G$.

In $H = Q_r$, $8\Delta = 8r$, $\varepsilon^{-\Delta} \geq 2^r$ and $n = 2^r$ ($\varepsilon \geq 1/2$), proves the goal as a private case.

Lemma $\varepsilon > 0$, $n \in \mathbb{N}$, $d > 0$. Suppose G graph with $N > 4d\varepsilon^{-d} n$ vertices and $\geq \varepsilon \frac{N^2}{2}$ edges. $\exists U \subseteq V(G)$ with $|U| \geq 2n$, s/t the fraction of d-sets $S \subseteq U$ with $|N^*(S)| < n$ is less then $(2d)^{-d}$.

pf. Choose d vertices randomly uniformly with repetitions - $\{v_1, \ldots, v_d\} = T$. As before $U = N^*(T)$. Then,

$$\mathbb{E}(|U|) = \sum_{v \in V(G)} \mathbb{P}(v \in U) = \sum_{v \in V(G)} \left(\frac{\deg(v)}{N}\right)^d \underset{\text{conv.}}{\geq} N \cdot \left(\frac{\sum \deg}{N^2}\right)^d \geq N \cdot \varepsilon^d.$$

Let x be the number of d-sets in U with $< n$ common neighbours - $\mathbb{E}(x) \leq \binom{N}{d}\left(\frac{n-1}{N}\right)^d$.

$$\mathbb{E}\left[|U|^d - \frac{\mathbb{E}[|U|]^d}{2\mathbb{E}[x]} x - \frac{\mathbb{E}[|U|]^d}{2}\right] \geq 0,$$ and therefore $\exists T$ s/t $|U|^d - \frac{\mathbb{E}[|U|]^d}{2\mathbb{E}[x]} x - \frac{\mathbb{E}[|U|]^d}{2} \geq 0$. In particular, $|U|^d \geq \frac{\mathbb{E}[|U|]^d}{2}$, so

$|U| \geq \left(\frac{1}{2}\right)^{1/d} \cdot N \cdot \varepsilon^d > 2n$. Also $x \leq \frac{2|U|^d \mathbb{E}[x]}{\mathbb{E}[|U|]^d} \leq 2|U|^d \cdot \binom{N}{d} \left(\frac{n-1}{N}\right)^d \cdot$

$$\cdot (N^d \varepsilon^{d^2})^{-1} \leq \frac{2|U|^d}{d!} \cdot \left(\frac{1}{4d}\right)^d \leq \binom{|U|}{d}\left(\frac{1}{2d}\right)^d$$

as we wanted to prove. Since $|U| > 2n \geq 2d$