V1401
25/1/15

Prob. Methods in Comb.

Lemma (Shearer) Let $X=(X_1,\dots,X_n):\Omega \to R_1\times\dots\times R_n$ be a r.v.. For each $I\subseteq[n]$, denote $X(I)=(X_i)_{i\in I}$. Suppose $\mathcal{G}\subseteq\mathcal{P}([n])$ is s.t. $\forall i\in[n]\ \exists\geq k$ distinct $G\in\mathcal{G}$ with $i\in G$. Then $H(X)\leq\frac{1}{k}\sum_{G\in\mathcal{G}} H(X(G))$.

pf. Induction on k, $k=1$ follows from $H(Y,Z)\leq H(Y)+H(Z)$. Suppose $k>1$. Case 1 If $[n]\in\mathcal{G}$ we can use the inductive assumption. Case 2 There are $G_1,G_2\in\mathcal{G}$ with $|G_1\cup G_2|>|G_1|,|G_2|$. It's enough to show $H(X(G_1\cup G_2))+H(X(G_1\cap G_2))\leq H(X(G_1))+H(X(G_2))$ so we can repeatedly replace G_1,G_2 by $G_1\cup G_2$, $G_1\cap G_2$ until $[n]\in\mathcal{G}$.
Denote $X(G_1\setminus G_2)=A$, $X(G_1\cap G_2)=B$, $X(G_2\setminus G_1)=C$.
We'll show $H(A,B,C)+H(B)\leq H(A,B)+H(B,C)$ or eq. $H(A|B,C)\leq H(A|B)$, property 3 of H that we still need to prove. This is $\sum_{a,b,c}\mathbb{P}(A=a,B=b,C=c)\Big[\log_2\frac{1}{\mathbb{P}(A=a,B=b,C=c)}+\log_2\frac{1}{\mathbb{P}(B=b)}-\log_2\frac{1}{\mathbb{P}(A=a,B=b)}-\log_2\frac{1}{\mathbb{P}(B=b,C=c)}\Big]\overset{②}{\leq}0$. The $[\,]$ term is $\log_2\underbrace{\frac{\mathbb{P}(A=a,B=b)\,\mathbb{P}(B=b,C=c)}{\mathbb{P}(B=b)\,\mathbb{P}(A=a,B=b,C=c)}}$ and by Jensen

$$\mathcal{C}=\sum \underbrace{w_{abc}\frac{\mathbb{P}(B=b,C=c)\,\mathbb{P}(A=a,B=b)}{\mathbb{P}(B=b)}}_{\alpha_{a,b,c}\text{ with }\sum_{a,b,c}\alpha_{a,b,c}=1}\log w_{abc}\geq\boxed{1/w_{abc}}$$

is implied by [using convexity of $x\overset{f}{\mapsto}x\log_2 x$]

$$\mathcal{C}\geq f\Big(\sum_{a,b,c}\alpha_{a,b,c}w_{a,b,c}\Big)=f\Big(\sum_{a,b,c}\mathbb{P}(A=a,B=b,C=c)\Big)=f(1)=0.$$

Ques. (Alon, 90) What n-vx. d-reg. graph has the most indep. sets?

Conj. (Alon, Kahn) $i(K_{d,d})^{\frac{n}{2d}}$ ($i(K_{d,d})=2^{d+1}-1$, so this is) $(2^{d+1}-1)^{\frac{n}{2d}}\approx(2^{d+1})^{\frac{n}{2d}}\approx 2^{\frac{n}{2}+\frac{n}{2d}}$.

Thm (Alon) $i(G)\leq 2^{\frac{n}{2}+\frac{Cn}{d^{0.1}}}$. **Thm** $i(G)\leq 2^{\frac{n}{2}+C\frac{n\sqrt{\log n}}{\sqrt{d}}}$, $\leq 2^{\frac{n}{2}+C\frac{\log d}{d}n}$.

Thm If G bipartite, $i(G)\leq i(K_{d,d})^{\frac{n}{2d}}$. **Thm** (Zhao) The bipartite assumption isn't neccesary.

pf. Let $X \sim$ uni(indep. sets of G), so $i(G) = 2^{H(X)}$. Suppose $V_0 \cup V_1$ is a bipartition. Think of X as the random vector $(X_v)_{v\in V}$. $H(X) = H(X(V_0)) + H(X(V_1) \mid X(V_0))$. For brevity we let $N_v = (X_w)_{w \sim v}$. $H(X(V_1) \mid X(V_0)) \leq \sum_{v\in V_1} H(X_v \mid X(V_0)) \leq \sum_{v \in V_1} H(X_v \mid N_v)$.

$H(X_v \mid N_v) = \mathbb{E}_A H(X_v \mid N_v = A) = \mathbb{P}(N_v = \emptyset)\cdot \log_2 2 + \mathbb{P}(N_v \neq \emptyset)\cdot 0 = \mathbb{P}(N_v = \emptyset) = p_v$.

We can think of N_v as $\left(\mathbb{1}_{\{N_v = A\}}\right)_A$. $H(N_v) = H(\mathbb{1}_{\{N_v=\emptyset\}}) + H(N_v \mid \mathbb{1}_{\{N_v = \emptyset\}}) = H(p_v) + (1-p_v) H(N_v \mid N_v \neq \emptyset) \leq H(p_v) + (1-p_v)\log_2(2^d - 1)$. By Shearer's we get $H(X(V_0)) \leq \frac{1}{d}\sum_{v\in V_1} H(N_v) \leq \frac{1}{d}\sum_{v\in V_1}[H(p_v) + (1-p_v)\log_2(2^d-1)]$

so $H(X(V_0)) + H(X(V_1)\mid X(V_0)) \leq \frac{1}{d}\sum_{v\in V_1}[d p_v + H(p_v) + (1-p_v)\log_2(2^d - 1)] \leq$ ($p_v \log_2 2^d$)

$\frac{n}{2d}\cdot \max_{p\in[0,1]}[p \log_2 2^d + H(p) + (1-p)\log_2(2^d-1)] =$ (achived for $\frac{2^d}{2^{d+1}-1}$)

$\frac{n}{2d}\log_2(2^{d+1}-1)$, so $i(G) \leq (2^{d+1}-1)^{\frac{n}{2d}}$.

Now what if G isn't bipartite? Take $G_2 = G \times \{0,1\}$ with edges only from 0 to 1 ($G_2 = G \times K_2$). Also $G_1 = G \sqcup G$.

$i(G_1) = i(G)^2$ so it's enough to show $i(G_1) \overset{?}{\leq} i(G_2) \overset{Kahn}{\leq} i(K_{d,d})^{\frac{2n}{2d}}$.

We inject $\{\text{Ind sets in } G_1\} \xrightarrow{\varphi} \{\text{" " " } G_2\}$. $\varphi(A,B)$ is defined as follows - $G[A \cup B]$ is bipartite (where $A \cap B \subseteq T$). Let (S,T) be some "canonical" 2-coloring of $G[A\cup B]$. Take $\varphi(A,B) = (A \triangle S)\times\{0\} \cup (B \triangle S)\times\{1\} = ((A\setminus S) \cup (B \cap S))\times\{0\} \cup ((B \setminus S) \cup ((A\setminus B)\cap S))\times\{1\}$

Same Color — Both in A — Both in B — Same Color

So it really is indep. If $\varphi(A,B) = \varphi(A',B')$ then $A\cup B = A' \cup B'$, $A \cap B = A' \cap B'$ so $S = S'$, and $A = A'$, $B = B'$.

$cr(G) =$ min # intersecting edges in a plane embedding of G.

<u>prop</u> $cr(G) \geq e(G) - 3v(G)$

pf By euler, G planar (on $v \geq 3$ vxs.) $\Rightarrow e(G) \leq 3v(G) - 6$.

After removing $cr(G)$ we have G' planar so we get the prop.

<u>Thm.</u> G on n vxs., $m \geq 4n$ edges $cr(G) \geq \frac{m^3}{n^2 \cdot 64}$.

<u>pf.</u> Draw G on the plane with $t = cr(G)$ crossings. Fix $p \in [0,1]$

and let $A \subseteq V(G)$ be random $\mathbb{P}(v \in A) = p$ idep. of other vxs.
Then $cr(G[A]) \geq e(G[A]) - 3|A|$. Taking $\mathbb{E}$ we get $\mathbb{E}\, cr(G[A]) \geq p^2 m - 3pn$. But, $cr(G[A]) \leq$ #crossing pairs that "survived" the random choice, so $\mathbb{E}[cr(G[A])] \leq cr(G) \cdot p^4$. Thus $cr(G) \geq \frac{m}{p^2} - \frac{3n}{p^3}$. Take $p = \frac{4n}{m}$ to get the bound.

Cor $|P| = n$ points in $\mathbb{R}^2$, $|L| = m$ distinct lines in $\mathbb{R}^2$, $I(P, L) = \#\{(p, \ell) \mid p \in P, \ell \in L, p \in \ell\}$, then $I(P, L) \leq C(m^{2/3} n^{2/3} + m + n)$, for some absolute const C (tight up to the value of C).

pf Define G with $V(G) = P$, $v_1 \sim v_2 \iff v_1, v_2$ are consecutive points on some $\ell \in L$. WLOG assume $\forall \ell \in L\ \exists p \in P, p \in \ell$. ($\#\geq 1$)
$e(G) = \sum_{\ell \in L} \#\{p \in P \mid p \in \ell\} - 1 = I(P, L) - m$. $cr(G) \leq \binom{m}{2} \leq \frac{m^2}{2}$. By prev. thm we get that either $I - m \leq 4n$ or $\frac{(I-m)^3}{64 n^2} \leq \frac{m^2}{2}$
so $I \leq \sqrt[3]{32 m^2 n^2} + m$

Conj (Erdős Szemerédi; '83) Let A be a set of n real numbers (can be integers). Let $A \cdot A = \{a \cdot b \mid a, b \in A\}$ and $A + A$ similarly defined. Then $\max\{|A \cdot A|, |A + A|\} \geq n^{-2+o(1)}$.

Thm $|A + B| \cdot |A \cdot B| \geq n^{5/2}$, for $|A| = |B| = n$.

pf. Let $P = \{(c, d) \mid c \in A + B, d \in A \cdot B\} = (A+B) \times (A \cdot B)$, and let $L = \{$"$y = a(x - a')$" for $a, a' \in A\}$. Then $P = |A+B| \cdot |A \cdot B|$, $L = n^2$. For any fixed $\ell \in L$ (defined by $a a'$), $\forall b \in B$, $a' + b \in A + B$, $a \cdot b \in AB$ so we have $\geq n$ pts. Hence, $I(P, L) \geq n^3$. By prev. thm we have $n^3 \leq C(n^{4/3} \cdot |P|^{2/3} + |P| + n^2)$. So either $|P|^{4/3} \geq n^{4/3}$ ✓, or $n^{4/3} |P|^{2/3} \geq \delta n^3$, $|P|^2 \geq \delta^3 n^5$, $|P| \geq \delta^{3/2} n^{5/2}$ ✓