W12D1
11/1/15

Prob. Methods in Comb.

Reminder: Yanson's inequality:

$\{B_i\}_{i\in I}$ events of the form $A_i \subseteq R$, $\mathbb{P}(\bigcap_{i\in I} \bar{B_i}) \approx e^{-\mu}$.

$X = \sum_{i\in I} \mathbb{1}_{B_i}$, then $X$ "$\sim$" $\mathrm{Pois}(\mu)$.

Bonferoni Inequalities

$A_1, \dots, A_n$ events in prob. space $\Omega$. Given $I \subseteq [n]$ let $A_I = \bigcap_{i\in I} A_i$. Given $\varepsilon \in \{0,1\}^n$, $A_\varepsilon = \bigcap_{i\in[n]} A_i^{\varepsilon_i}$ where $A^0 = \bar{A}$, $A^1 = A$.

$\underbrace{\mathbb{P}(A_{\{0,\dots,0\}}) = \sum_{r=0}^{n} \sum_{|I|=r} (-1)^r \mathbb{P}(A_I)}_{\text{inc.-ex.}}$. Bonferoni - $\forall s \geq 0$,

$\sum_{r=0}^{2s+1} \dots \leq \mathbb{P}(A_{\{0,\dots,0\}}) \leq \sum_{r=0}^{2s} \sum_{|I|=r} (-1)^r \mathbb{P}(A_I)$.

Brun's Sieve

$n \to \infty$, $\Omega(n)$ prob. space & $A_1(n), \dots, A_{m(n)}(n)$ events in it.

$X_{(n)} = \sum_{i=1}^{m(n)} \mathbb{1}_{A_i(n)}$, $S^{(r)} = S^{(r)}(n) = \sum_{|I|=r} \mathbb{P}(A_I(n))$. Suppose $\exists \mu$ const. s.t. $\forall r \geq 1$, $\lim_{n\to\infty} S^{(r)}(n) = \frac{\mu^r}{r!}$ (in particular $\mathbb{E}[X] \to \mu$). Then for every fixed $t$,

$\lim_{n\to\infty} \mathbb{P}(X = t) = \frac{\mu^t}{t!} e^{-\mu} = \mathbb{P}(\mathrm{Pois}(\mu) = t)$.

pf. For $t=0$, fix $\delta > 0$. First choose $S_0$ s.t. $\forall s > s_0$, ~~looking at the~~ $\left|\sum_{r=0}^{s} \frac{\mu^r}{r!}(-1)^r - e^{-\mu}\right| < \delta/2$. Choose $n_0$ s.t. $\forall n > n_0$ $\left|S^{(r)}(n) - \frac{\mu^r}{r!}\right| < \frac{\delta}{2(s_0+2)}$ for all $r \leq 2s_0+1$. By Bonferoni, $e^{-\mu} - \delta \leq \sum_{r=0}^{2s_0+1} (-1)^r \frac{\mu^r}{r!} - \frac{\delta}{2} \leq \sum_{r=0}^{2s_0+1} S^{(r)}(n) \leq \mathbb{P}(X=0) \leq \sum_{r=0}^{2s_0} (-1)^r S^{(r)}(n) \leq \sum_{r=0}^{2s_0} \frac{(-1)^r \mu^r}{r!} + \frac{\delta}{2} \leq e^{-\mu} + \delta$. For $t \neq 0$ fix $T \subseteq I$ of size $t$. $\mathbb{P}$(Only $A_i$ with $i \in T$ hold) can be estimated using the prev. methods using $X_T$ = # events $\notin T$ which hold, and suming over $|T| = t$ we get $\frac{\mu^t}{t!} e^{-\mu} \sum_{r=0}^{\infty} \binom{r+t}{t} \frac{\mu^{r+t}}{(r+t)!}$

Example: Theshold func. for $\forall v\, x \in$ Copy of $K_3$ -

Thm Let $p = p(n)$, $\mu = \mu(n)$ s.t. $e^{-\mu} = \frac{c}{n}$ ($c > 0$ fixed), $\mu = \binom{n-1}{2} c^3$ (so $\mu \simeq \log n$, $p \simeq n^{-2/3} (\log n)^{1/3}$). Then $\mathbb{P}(G(n,p)$ has the property$) = e^{-c}$.

pf. $G \sim G(n,p)$. $B_{xyz} = \{xyz$ induce a triangle$\}$. $A_x = \bigcap_{y,z \neq x} \overline{B_{xyz}}$.

Apply Yanson with $\mu = \mathbb{E}[\# B_{xyz}$ that hold$] = p^3 \binom{n-1}{2} = \mu$ (to estimate; $x$ fixed) and

$\Delta = n^{-1/3+o(1)} = o(1)$, so $\mathbb{P}(A_x) \leq e^{-\mu - o(1)} = (1+o(1)) e^{-\mu}$, and by

Harri's inequality / FKG $\mathbb{P}(A_x) \geq (1-p^3)^{\binom{n-1}{2}} \geq (1-o(1))e^{-\mu}$. Now put $X = \sum_{x\in V} \mathbb{1}_{A_x}$. $\mathbb{E}[X] = S^{(1)} = (1-o(1))e^{-\mu}\cdot n = (1-o(1))c$. We want $\mathbb{P}(X=0) \to e^{-c}$ so

it's enough $S^{(r)} \longrightarrow \frac{c^r}{r!}$. Let $|T|=r$, $A_T = \bigcap_{x\in T} A_x = \bigcap_{\substack{S\cap T\neq\emptyset \\ |S|=3}} \overline{B_S}$, $\mu_T = \left[r\binom{n-1}{2} - O(n)\right]p^3$,

$\Delta = O(p^5 \cdot r \cdot n^3) = o(1)$, so $\mathbb{P}(A_T) = e^{-\mu r}(1-o(1))$ by Janson-FKG. So

$S^{(r)} = \binom{n}{r}e^{-\mu r}(1-o(1)) = \frac{n^r e^{-\mu r}}{r!} = \frac{(ne^{-\mu})^r}{r!}(1-o(1)) = \frac{c^r}{r!}(1-o(1))$.

## Large Deviation Ineq. for sums of "Almost Indep." r.v.s

$\Omega$ fin. $R \subseteq \Omega$ randomly $\mathbb{P}(x\in R) = p_x$ indep. $B_i =$ "$A_i \subseteq R$".

$i \sim j \Leftrightarrow i\neq j, A_i \cap A_j \neq \emptyset$. We call $J \subseteq I$ a disjoint family if $J$ is $\sim$-indep. Maximal disjoint family if $J$ can't be extended.

Thm. $\forall$ integers $s$, $\mu = \sum_{i\in I} \mathbb{P}(B_i)$, $\mathbb{P}(\exists |J| = s$ disj. family$) \leq \frac{\mu^s}{s!} \leq \left(\frac{e\mu}{s}\right)^s$. In particular, if $s = (e+\delta)\mu$, this $\mathbb{P} \leq (1-\rho(\delta))^{\mu}$.

pf. $\mathcal{A}^s = \{|J| \subseteq I \ \sim\text{-indep}\}$. $\mathcal{A}_{ord} \{(j_1,\dots,j_s) \in I^s \mid \{j_1,\dots,j_s\} \sim\text{-ind.}\}$.

$\mathbb{P}(\exists |J| = s \text{ disj}) \leq \sum_{J\in\mathcal{A}} \mathbb{P}(\bigwedge_{j\in J} B_j) = \sum_{J\in\mathcal{A}} \prod_{j\in J} \mathbb{P}(B_j) = \frac{1}{s!}\sum_{J\in\mathcal{A}_{ord}} \mathbb{P}(B_{j_1})\dots\mathbb{P}(B_{j_s}) \leq$

$\frac{1}{s!}\sum_{J\in I^s} \mathbb{P}(B_{j_1})\dots\mathbb{P}(B_{j_s}) = \frac{1}{s!}\left(\sum_{j\in I}\mathbb{P}(B_j)\right)^s = \frac{\mu^s}{s!}$.

## Illustration - LBs for off-diag. Ramsey nums. (Krivelevich)

Fix $\ell \geq 3$. $R(\ell,k) = \max\{n \mid \exists G$ on $n$ vxs. $G \not\supseteq K_\ell$, $\alpha(G) < k\}$.

prop. $R(\ell,k) \geq C_\ell \left(\frac{k}{\log k}\right)^{\frac{\ell-1}{2}}$ for some $C_\ell > 0$.

For $n$ above $\ell$ $p = c' n^{-\frac{2}{\ell-1}}$, $n^\ell p^{\binom{\ell}{2}} = \Theta(n^2 p)$. For $S \subseteq V(G)$

$X_S = e_{G(n,p)}(S)$, $Y_S = \#$ copies of $K_\ell$ in $G(n,p)$ with $\geq 2$ vxs in $S$, $Z_S = \#$ max edge disjoint such copies. $A_S =$ "$X_S > \binom{\ell}{2}Z_S$".

If $\bigcap_{|S|=k} A_S$ holds, $G(n,p)$ witnesses $R(k,\ell) \geq n$. Let $H$ be a family in the def. of $Z_S$. Delete from $G$ edges in those cliques-
$e_G(S) \geq X_S - \binom{\ell}{2}Z_S > 0$ for any $S \in \binom{[n]}{[k]}$ in the resulting graph, and it's also $K_\ell$-free. Now, for fixed $S \in \binom{[n]}{[k]}$, $\mathbb{P}(\overline{A_S}) \leq \mathbb{P}(X_S \leq \mathbb{E}X_S/2) +$ (chernoff)

$\mathbb{P}(\binom{\ell}{2}Z_S \geq \mathbb{E}X_S/2)$ (large dev.). $\mathbb{E}[X_S] = \binom{k}{2}p$, so $\leq e^{-\binom{k}{2}p/8}$. For $2^{nd}$ part,

$\mathbb{E}(Z_S) \leq \mathbb{E}(Y_S) =: \mu_S \leq \binom{k}{2}\binom{n-2}{\ell-2}p^{\binom{\ell}{2}}$, $\mu_S \geq \binom{k}{2}\binom{n-k}{\ell-2}p^{\binom{\ell}{2}}$ but $k = o(n)$ so there both $\Theta(\binom{k}{2}n^{\ell-2}p^{\binom{\ell}{2}})$. By lemma $\mathbb{P}(Z_S \geq 5\mu_S) \leq (e/5)^{5\mu_S}$. So

$\frac{c_\ell}{c'^{\binom{\ell}{2}-1}} \leq \frac{\mathbb{E}[X_S]}{\mu_S} \leq \frac{\binom{k}{2} p}{\binom{k}{2} n^{\ell-2} p^{\binom{\ell}{2}}} = \frac{1}{c_\ell (c')^{\binom{\ell}{2}-1}}$ for ~~small enough~~ some $c_\ell > 0$. For

small enough $c'$, $10\binom{\ell}{2} \leq \frac{\mathbb{E}[X_S]}{\mu_S} \leq D = \Theta(1)$. With this choice of $c'$, $\mathbb{P}(\binom{\ell}{2} Z_S \geq \mathbb{E}X_S/2) \leq \mathbb{P}(Z_S \geq 5\mu_S) \leq (e/5)^{\frac{\mathbb{E}[X_S]}{D}} = e^{-\tilde{c}\binom{k}{2}p}$.

Now we know $\mathbb{P}(\bar{A}_S) \leq 2e^{-\tilde{c}\binom{k}{2}p}$. So, $\mathbb{P}(\bigwedge_{|S|=k} A_S) \geq 1 - \sum_{|S|=k} \mathbb{P}(\bar{A}_S) \geq 1 - \binom{n}{k} e^{-\tilde{c}\binom{k}{2}p} \geq 1 - \left(\frac{en}{k} e^{-\tilde{c}\frac{k-1}{2}p}\right)^k$. (2) follows from $pk \geq \frac{2\ell}{\tilde{c}} \log k \iff p \geq \frac{2\ell}{\tilde{c}k} \log k \stackrel{(?)}{=} \frac{2\ell}{\tilde{c}} \left(\frac{n}{c}\right)^{-\frac{2}{\ell-1}} = \frac{2\ell}{\tilde{c}\, c^{-\frac{2}{\ell-1}}} \frac{p}{c'}$ and we can just choose $c$ sufficiently small.

Pois. concentration of measure - $\mathbb{P}(P \leq (1-\varepsilon)\mu) \leq e^{\varepsilon^2\mu/2}$, $\mathbb{P}(P \geq (1+\varepsilon)\mu) \leq \left[e^{\varepsilon}(1+\varepsilon)^{-(1+\varepsilon)}\right]^{\mu}$. Define $\mathbb{N} \ni s \geq 0$, $\mu_s = \min\{\sum_{i \in J} \mathbb{P}(B_i) \mid |J| = s, J \subseteq I\}$.

Letting $\nu = \max_{i \in I} \sum_{j \sim i} \mathbb{P}(B_j)$, then $\mu_s \geq \mu - s\nu$.

<u>Lemma</u> $\mathbb{P}(\text{max. disj. family of size } s) \leq \frac{\mu^s}{s!} e^{-\mu_s + \Delta/2}$