Prob. Methods in Comb.

Talagrand's inequality - Ω_i prob. space, $\Omega = \prod_{i=1}^{n} \Omega_i$ with product measure.

Thm (Talagrand) For any $A \subseteq \Omega$, $x \in \Omega$ random, $t \geq 0$,

$$\mathbb{P}(x \in A)\,\mathbb{P}(\rho(x, A) \geq t) \leq e^{-t^2/4} \text{ where } \rho(x,A) = \sup_{\|\alpha\|_2 = 1} \min_{y \in A} \overbrace{\sum_{i : x_i \neq y_i} \alpha_i}^{d_\alpha(x,y)}.$$

Example longest increasing subsequence of a random permutation.

In HW- $LIS(\sigma) = \Theta(\sqrt{n})$ a. a. s. when $\sigma \in S_n$ u. a. r.

We show a. a. s. $LIS(\sigma)$ belongs to an interval of length $\omega(n) \cdot n^{1/4}$ for any $\omega(n) \to \infty$.

Fix $f : \mathbb{N} \to \mathbb{N}$. h is <u>f-certifiable</u> if whenever $h(x) \geq s$ $\exists I_x \subseteq [n]$, $|I_x| \leq f(s)$ and $\forall y \in \Omega$ s.t. $x|_{I_x} = y|_{I_x} \to h(y) \geq s$.

Ex $\Omega = \{0,1\}^{\binom{n}{2}}$, $h = \#\Delta$ in G, h is f-cert. for $f(s) = 3s$.

Ex If $f(s) = n$ any h is f-cert.

Thm Let $h : \Omega \to \mathbb{R}$ be f-cert. Lipschitz function.

Let $X \in \Omega$ random, look at $h(X)$. Then $\forall b, t$ -

$$\mathbb{P}(h(X) \geq b)\,\mathbb{P}(h(X) \leq b - t\sqrt{f(b)}) \leq e^{-t^2/4}.$$

pf. $A = \{y \in \Omega \mid h(y) < b - t\sqrt{f(b)}\}$. Suppose $h(x) \geq b$ - it's enough to show $\rho(x, A) \geq t$ and apply Talagrand. If $h(x) \geq b$ $\exists I_x \ldots$

Set $\alpha_i := \begin{cases} |I_x|^{-1/2} & i \in I_x \\ 0 & \text{o/w} \end{cases}$ so that $\|\alpha\|_2 = 1$. So, $d_\alpha(x, A) < t$ and $\exists y \in A$ with $d_\alpha(x,y) < t$. Define $x'_i = \begin{cases} x_i & i \in I_x \\ y_i & \text{o/w} \end{cases}$,

Then $h(x') \geq b$ by def. of I_x $\to d_\alpha(y, x) = \frac{1}{|I_x|^{1/2}} \#\{x_i \neq y_i\}$. So,

$d_{\#}(x', y) = \{i \in I_x \mid x_i \neq y_i\} < t \cdot |I_x|^{1/2} \leq t\sqrt{f(b)}$ but then

$h(y) \geq h(x') - d_h(x', y) > b - t\sqrt{f(b)}$ - but $y \in A$ so $h(y) < b - t\sqrt{f(b)}$ in contradiction.

For our setting note that replacing random $\sigma \in S_n$ with $x \in [0,1]^n$ doesn't change the dist. of $LIS(x)$. It's Lipschitz and $(s \mapsto s)$ certifiable. Let b = median of $LIS(x) = \Theta(\sqrt{n})$. Then

then $\overset{\frac{1}{2}}{\geq} \mathbb{P}(LIS(x) \overset{\wedge}{\geq} m)\, \overset{(*)}{\mathbb{P}}(LIS(x) \leq m - t\sqrt{m}) \leq e^{-t^2/4}$. For the upper bound $m = b - t\sqrt{b}$ we get $\mathbb{P}(|LIS(x) - m| \leq t\sqrt{m}) \leq 4e^{-t^2/4}$.

Bounded differences only gives $2e^{-2t^2} \leq \mathbb{P}(|LIS - \mu| \leq t\sqrt{n})$.

Sketch of pf. of Talagrand:

$$\mathbb{P}(\rho(x,A) \geq t) = \mathbb{P}\left(e^{\frac{\rho^2(x,A)}{4}} \geq e^{t^2/4}\right) \overset{\text{Markov}}{\leq} e^{-\frac{t^2}{4}} \mathbb{E}\left[e^{\rho^2(x,A)/4}\right]$$

$(*) \longrightarrow \overset{\wedge}{1/\mathbb{P}(x \in A)}$

(*) Can be proved inductively on n.

Today: Yanson's inequality

Motivation We want to estimate $\mathbb{P}(\bigcap_{i\in I} \overline{B_i})$. ~~Jensen~~ Yanson gives an upper bound.

Note - If B_i where mutually ind. was just $\prod_{i\in I} 1 - \mathbb{P}(B_i) \approx e^{-\mathbb{E}(\# B_i \text{ that hold})}$ for $\mathbb{P}(B) \ll 1$. Jensen - under certain assumption this is close to the truth.

Setting Ω fin. For each $x \in \Omega$ fix $p_x \in [0,1]$ and let $R \subseteq \Omega$ random, where $\forall x\ \mathbb{P}(x \in R) = p_x$ indep. of $x' \neq x \in \Omega$. Fix $A_i^{\in \Omega}$ and let $B_i = \{A_i \subseteq R\}$. For $i,j \in I$ write $i \sim j$ iff $i \neq j$ but $A_i \cap A_j \neq \emptyset$. Define $M = \prod_{i\in I}(1 - \mathbb{P}(B_i))$, $\mu = \mathbb{E}[\# B_i \text{ that hold}] = \sum_{i\in I} \mathbb{P}(B_i)$, $\Delta = \mathbb{E}[(i,j) \in I^2 \mid i \sim j,\ B_i, B_j \text{ hold}] = \sum_{i\sim j} \mathbb{P}(B_i \cap B_j) = \sum_{i \sim j} \prod_{x \in A_i \cup A_j} p_x$.

Yanson In the above notation, $M \overset{\text{FKG}}{\leq} \mathbb{P}(\bigcap_{i\in I} \overline{B_i}) \overset{(*)}{\leq} e^{-\mu + \Delta/2}$. If moreover, $\forall i\ \mathbb{P}(B_i) \leq \varepsilon$, $\mathbb{P}(\bigcap_{i\in I} \overline{B_i}) \leq M \cdot e^{\frac{1}{1-\varepsilon} \cdot \frac{\Delta}{2}}$.

comment If $\varepsilon, \Delta = o(1)$ then $\mathbb{P}(\bigcap_{i\in I} \overline{B_i}) = (1 + o(1))e^{-\mu} = (1 + o(1))M$.
If $\Delta \geq 2\mu$, (*) is useless but Thm (Extended ~~Jensen~~ Yanson) If $\Delta \geq \mu$, $\mathbb{P}(\bigcap_{i\in I} \overline{B_i}) \leq e^{-\mu^2/2\Delta}$.

~~Jensen~~
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↓
Ex. Yanson

pf. Given $J \subseteq I$ define $\mu(J), \Delta(J)$ like we did with I. Let $q \in [0,1]$ and J be a random subset of I, keeping each element with prob. q. $\mathbb{E}\left[-\mu(J) + \frac{\Delta(J)}{2}\right] = -q\mu + \frac{1}{2}q^2\Delta \overset{q = \frac{\mu}{\Delta}}{=} -\frac{\mu^2}{2\Delta}$.
Hence $\exists J$ with $-\mu(J) + \Delta(J)/2 \leq \frac{-\mu^2}{2\Delta}$ so we get the bound by applying (regular) Yanson.

pf. Assume $I=\{1,\dots,m\}$. $\mathbb{P}(\bigcap_{i=1}^{m}\overline{B_i})=\prod_{i=1}^{m}\overbrace{\mathbb{P}(\overline{B_i}\mid\bigcap_{j=1}^{i-1}\overline{B_j})}^{1-\mathbb{P}_{\dots}}$. We claim

Yanson claim $\mathbb{P}(B_i)-\sum_{j\sim i}\mathbb{P}(B_i\cap B_j)\leq\mathbb{P}(B_i\mid\bigcap_{j=1}^{i-1}\overline{B_i})\overset{FKG)}{\leq}\mathbb{P}(B_i)$ Claim $\Rightarrow\mathbb{P}(\bigcap_{i=1}^{m}\overline{B_i})\geq\prod_{i=1}^{m}(1-\mathbb{P}(B_i)+\sum_{j\sim i}\mathbb{P}(B_i\cap B_j))$

which is $\leq\prod_{i=1}^{m}e^{-\mathbb{P}(B_i)+\sum_{i\sim j}\mathbb{P}(B_i\cap B_j)}=e^{-\mu+\Delta/2}$ but also

(if $\forall i\ \mathbb{P}(B_i)\leq\varepsilon$) $\leq\prod_{i=1}^{m}[1-\mathbb{P}(B_i)]\left[1+\frac{\sum_{i\sim j}\mathbb{P}(B_i\cap B_j)}{1-\varepsilon}\right]\leq M\cdot e^{\frac{\Delta}{2(1-\varepsilon)}}$.

pf. claim $D_i=\bigcap_{j=1}^{d}\overline{B_j}$, $U_i=\bigcap_{j=d+1}^{i-1}\overline{B_j}$, $\mathbb{P}(B_j\mid D_i\cap U_i)\geq\mathbb{P}(B_i\cap D_i\mid U_i)=$

(we assume $\{j\sim i\}=\{1,\dots,d\}$)... $=\overset{\mathbb{P}(B_i)}{\overset{\shortparallel}{\mathbb{P}(B_i\mid U_i)}}\mathbb{P}(D_i\mid B_i\cap U_i)$.

$\mathbb{P}(D_i\mid B_i\cap U_i)=1-\mathbb{P}(\bigcup_{j=1}^{d}B_i\mid B_i\cap U_i)\geq1-\sum_{j=1}^{d}\mathbb{P}(B_j\mid B_i\cap U_i)\underset{②}{\geq}1-\sum_{j=1}^{d}\mathbb{P}(B_j\mid B_i)$

It's enough to prove (*) term by term.

$\mathbb{P}(B_j\mid B_i\cap U_i)\overset{\text{condition on } A_i\subseteq R \text{ - ie } p_x=1 \text{ for } x\in A_i}{=}\mathbb{P}^*(B_j\mid U_i)\leq\mathbb{P}^*(B_j)=\mathbb{P}(B_j\mid B_i)$.

Ex. Clique num of $G(n,1/2)$. Let $f(k)=\binom{n}{k}2^{-\binom{k}{2}}$. Def. $k=\min\{k\mid f(k)<1\}$.

$\frac{f(k_0+1)}{f(k_0)}\approx n^{-1+o(1)}$ so a. e. s. $\omega(G)<k_0+1$. $k_1=k_0-4$, $f(k_1)=n^{4+o(1)}$

and we proved $\mathbb{P}(\omega(G)<k_1)\leq e^{-c\frac{n^2}{\log^8 n}}$. We can now improve to $\log^4 n$. $I=\binom{[n]}{k}$. Given $S\in I$ define $B_S=\{G[S]\text{ clique}\}$.

Then $\mathbb{P}(\omega(G)\geq k_1)=\mathbb{P}(\bigcap_{S\in I}\overline{B_S})$. $\mu=f(k_1)\geq n^{3+o(1)}$, $\Delta=\binom{n}{k_1}2^{-\binom{k_1}{2}}\sum_{s=2}^{k_1-1}\underbrace{\binom{k_1}{s}\binom{n-k_1}{k_1-s}2^{-\binom{k_1}{2}+\binom{s}{2}}}_{g(s)}=$

$(1+o(1))\frac{f^2(k_1)k_1^4}{n^2}=(1+o(1))\frac{\mu^2k_1^4}{n^2}\gg\mu$.

By EYI $\mathbb{P}(\omega(G)\geq k_1)\leq e^{-\frac{\mu^2}{2\Delta}}\leq e^{-c\frac{n^2}{k_1^4}}=e^{-c'\frac{n^2}{\log^4 n}}$.

Ex. Triangles in $G(n,p)$ - What is $\mathbb{P}(G\text{ is }\Delta\text{-free})$? For large p this in $\approx\mathbb{P}(G$ bipartite). But what if $p=\frac{c}{n}$? X - num of Δ in G. $\mathbb{E}[X]=(1+o(1))\frac{c^3}{6}$ [In fact, $X\approx\mathrm{Pois}(\frac{c^3}{n})$].

claim $\mathbb{P}(X=0)\approx e^{c^3/6}$.

pf. Yanson with $I=\binom{[n]}{3}$, $B_S=\{G[S]\cong K_3\}$. Then $\mu=\binom{n}{3}p^3=(1+o(1))\frac{c^3}{6}$.

$\Delta=\binom{n}{3}(n-3)\cdot3\cdot p^5=\Theta(n^4p^5)=\Theta(1/n)=o(1)$. By ~~Yanson~~ Yanson & FKG

$(1+o(1))e^{-\frac{c^3}{6}}=(1-p^3)^{\binom{n}{3}}\leq\mathbb{P}(X=0)\leq e^{-\mu+\Delta/2}=(1+o(1))e^{\frac{c^3}{6}}$.