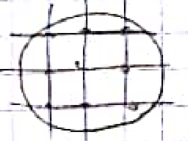
Lecture 5:

11/4/2018



The circle problem

$$N(R) := \# \mathbb{Z}^2 \cap Ball(0,R)$$

Showed: $N(R) = \pi R^2 + O(R)$ length of bdry.

Problem Good upper bound for

$$|P(R)| := |N(R) - \pi R^2|$$

Conj

$$P(R) = O(R^{1/2 + o(1)})$$

Goals:

1) $P(R) = O(R^{2/3})$ ← 1905 Sierpinski, Voronoi Van der Corput 1920

2) $P(R) = \Omega(R^{1/2})$ i.e $\exists c > 0$ $\exists$ arbitrarily large R's s.t $|P(R)| > cR^{1/2}$

Oscillatory Integrals

$$I(\lambda) := \int_{\mathbb{R}} A(x) e^{i\lambda \Phi(x)} dx \qquad \lambda \to \infty$$

$A(x) \in$ Compactly supported – "Amplitude"

$\Phi(x)$ real valued smooth – "phase"

(**Goal:** The Fourier transform of the unit ball

$$\int_{x^2+y^2 \le 1} e^{2\pi i (ax+by)} dxdy, \quad (a,b) \longrightarrow \infty$$)

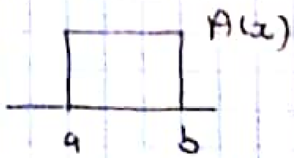
Trivial Bound:

$$|I(\lambda)| \le \int |A(x)| dx$$

Want: $I(\lambda) \longrightarrow 0$ as $\lambda \longrightarrow \infty$

Example:

$A(x) = 1_{[a,b]}$ $\quad \displaystyle\int_a^b e^{i\lambda \Phi(x)} dx$



Example:

$$J_0(\lambda) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\lambda \overbrace{\sin(t)}^{\phi(t)}}\,dt \quad \text{- Bessels function}$$

$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{z}{2}\right)^m$$

Thm:

$$J_0(\lambda) \underset{\lambda\to\infty}{<<} \frac{1}{\sqrt{\lambda}}$$

From the summation representation we get $J_0(z) \underset{z\to 0}{\sim} 1$.

Principle of non-stationary phase

If $A(x) \in C_c^\infty(\mathbb{R})$ is smooth, compactly supported &

$\phi' \neq 0$ on support (A)

then $I(\lambda) << \frac{1}{\lambda^N} \quad \forall N \geq 1$

Proof:

Integrate by parts.

$$\int A(x) e^{i\lambda\phi(x)} = \int \frac{A(x)}{i\lambda\phi'(x)} \overbrace{i\lambda\phi'(x) e^{i\lambda\phi(x)}}^{\frac{d}{dx}\left(e^{i\lambda\phi}\right)}\,dx =$$

$$= \underbrace{\frac{A(x)}{i\lambda\phi'(x)} e^{i\lambda\phi(x)}\Big|_{-\infty}^{\infty}}_{0} - \frac{1}{i\lambda}\int_{-\infty}^{\infty} \underbrace{\frac{d}{dx}\left\{\frac{A(x)}{\phi'(x)}\right\}}_{A_1(x)} e^{i\lambda\phi}\,dx$$

$$|I(\lambda)| \leq \frac{1}{\lambda}\left|\int A_1(x) e^{i\lambda\phi(x)}\,dx\right| \leq \frac{1}{\lambda}\int |A_1(x)|\,dx << \frac{1}{\lambda};$$

Now iterate N times.

$\blacksquare$

Van der Corput's Lemma 1:

Assume ϕ real, smooth, $\phi' \neq 0$ on $[a,b]$ & ϕ' monotonic.

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\,dx, \quad \text{Then}$$

$$|I(\lambda)| \leq \frac{4}{\min\limits_{x\in[a,b]} |\phi'(x)|}\cdot\frac{1}{\lambda}$$

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Proof:

Integrate by parts.

$$\int_a^b e^{i\lambda\phi(x)} = \int_a^b \frac{i\lambda\phi' e^{i\lambda\phi}}{i\lambda\phi'} = \int_a^b \frac{1}{i\lambda\phi'} \cdot \frac{d}{dx}\left\{e^{i\lambda\phi(x)}\right\} dx =$$

$$= \frac{1}{i\lambda\phi'(x)} e^{i\lambda\phi(x)}\Big|_a^b - \int_a^b \frac{d}{dx}\left\{\frac{1}{i\lambda\phi'}\right\} e^{i\lambda\phi} dx$$

Bdry term

$$\left|\frac{1}{i\lambda}\left\{\frac{e^{i\lambda\phi(b)}}{\phi'(b)} - \frac{e^{i\lambda\phi(a)}}{\phi'(a)}\right\}\right| \leq \frac{1}{\lambda} \cdot \frac{2}{\min_{[a,b]}|\phi'|}$$

$$\left|\int_a^b \frac{d}{dx}\left\{\frac{1}{i\lambda\phi'}\right\} e^{i\lambda\phi} dx\right| \leq \frac{1}{\lambda}\int_a^b \left|\frac{d}{dx}\left\{\frac{1}{\phi'}\right\}\right| \cdot 1\, dx$$

ϕ' monotone $\Rightarrow \frac{1}{\phi'}$ monotone $\Rightarrow \frac{d}{dx}\left\{\frac{1}{\phi'}\right\}$ has a fixed sign in $[a,b]$ $\Rightarrow \int |*| = \left|\int *\right|$

$$\int_a^b \left|\frac{d}{dx}\left\{\frac{1}{\phi'}\right\}\right| dx = \left|\int_a^b \frac{d}{dx}\left\{\frac{1}{\phi'}\right\} dx\right| = \left|\frac{1}{\phi'(b)} - \frac{1}{\phi'(a)}\right| \leq \frac{2}{\min_{[a,b]} \phi'}$$

$$\Rightarrow |I(\lambda)| \leq \frac{4}{\min_{[a,b]}|\phi'|} \cdot \frac{1}{\lambda}, \quad \lambda > 0.$$

$\blacksquare$

Van der Corput's Lemma 2:

Assume ϕ real, smooth & $\phi'' \neq 0$ on $[a,b]$

then

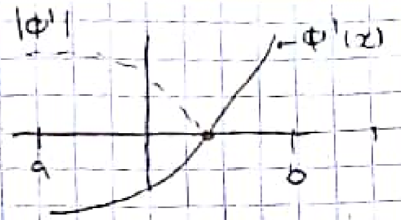
$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)} dx \leq \frac{8}{\left(\min_{[a,b]}|\phi''|\right)^{1/2}} \cdot \frac{1}{\sqrt{\lambda}}$$

More generally, if $k \geq 2$ & $\phi^{(k)}(x) \neq 0$ on $[a,b]$ then:

$$|I(\lambda)| \leq \frac{C_k}{\left(\min_{[a,b]}|\phi^{(k)}|\right)^{1/k}} \cdot \frac{1}{\lambda^{1/k}}$$

$C_k > 0$ absolute constant.

Proof of VdC 2:

Lets draw ϕ' , let c be such that

$|\phi'(c)|$ is minimal.

Two cases: $a<c<b$ or $c=a,b$

Assume $a<c<b$

Rescale: replace ϕ by $\phi_1 := \frac{\phi}{\min|\phi''|}$; now $|\phi_1''| \geq 1$

$$\int e^{i\lambda\phi} = \int e^{i(\min|\phi''|\cdot\lambda)\frac{\phi}{\min|\phi''|}} = \hat{I}(\min|\phi''|\cdot\lambda)$$

Take $\delta>0$ small; then outside of $(c-\delta, c+\delta)$ 

we have $|\phi'(t)| \geq \delta$

$$\phi_1'(t) = \phi_1'(t) - \underbrace{\phi_1'(c)}_{0} = \int_c^t \phi_1''(x)\,dx.$$

$$(t>c+\delta) \quad \phi_1'' \geq 1 \implies \phi_1'(t) \geq \int_c^t 1 = t-c \geq \delta$$

use VdC1 on $[a, c-\delta]$ & $[c+\delta, b]$

$$\left|\int_a^{c-\delta} e^{i\lambda\phi_1(x)}\,dx\right| \leq \frac{4}{\delta}\cdot\frac{1}{\lambda}, \quad \left|\int_{c+\delta}^{b} e^{i\lambda\phi_1(x)}\,dx\right| \leq \frac{4}{\delta\lambda}$$

$$\left|\int_{c-\delta}^{c+\delta} e^{i\lambda\phi_1(x)}\,dx\right| \underset{\text{trivial}}{\leq} \int_{c-\delta}^{c+\delta} 1 = 2\delta$$

$$\left|\underset{[\phi_1]}{I(\lambda)}\right| = \left|\int_a^b\right| = \left|\int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^{b}\right| \leq \frac{4}{\delta\lambda} + 2\delta + \frac{4}{\delta\lambda}$$

Choose δ to minimize RHS $\implies I(\lambda) \leq \frac{8}{\sqrt{\lambda}}$ + this is for ϕ_1.

Corollary

$A \in C^1[a,b]$, ϕ real, smooth, $\phi^{(k)} \neq 0$ on $[a,b]$ (& if $k=1$, ϕ' is monotone

Then
$$\left|\int_a^b A(x)e^{i\lambda\phi(x)}\,dx\right| \leq \frac{C_k}{(\min_{[a,b]}|\phi^{(k)}|)^{1/k}}\left(|A(b)| + \int_a^b |A'(x)|\,dx\right)\frac{1}{\lambda^{1/k}}$$

Proof:

Integrate by parts.

$$\int_a^b A(x) e^{i\lambda\phi(x)} = \int_a^b A(x) \frac{d}{dx} J_\lambda \, dx = A(x) J_\lambda(x)\Big|_a^b - \int_a^b A'(x) J_\lambda(x) \, dx$$

$$J_\lambda(t) := \int_a^t e^{i\lambda\phi(x)} dx .$$

$$\leq A(b) \underbrace{\left|\int_a^b e^{i\lambda\Phi(x)} dx\right|} + \int_a^b |A'(x)| \frac{C_k}{\min\limits_{[a,b]} \phi^{(k)}} \frac{1}{\lambda^{1/k}} dx$$

$VdC_k \to \ll \dfrac{1}{\lambda^{1/k}}$

∎

Main Application – the Fourier-transform of the unit disk.

$$\hat{\chi}(\xi) := \int_{|x| \leq 1} e^{ix\cdot\xi} dx , \quad \xi \in \mathbb{R}^2$$

$$= \int_{\mathbb{R}^2} 1_{B(0,1)}(x) e^{ix\xi} dx$$

Thm:

$$|\hat{\chi}(\xi)| \ll \frac{1}{|\xi|^{3/2}} \qquad |\xi| \geq 1$$

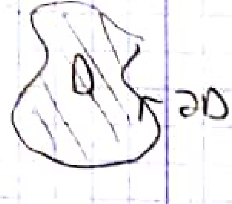
Baby example: 1 Dimensional case = Fourier transform of unit interval.

$$\int_{\substack{|x| \leq 1 \\ x \in \mathbb{R}^1}} e^{ix\xi} dx = \int_{-1}^{1} e^{ix\xi} dx \overset{\xi \neq 0}{=} \frac{e^{ix\xi}}{i\xi}\Big|_{-1}^{1} = \frac{e^{i\xi} - e^{-i\xi}}{i\xi} = \frac{2\sin\xi}{\xi}$$

so get bd $\ll \dfrac{1}{|\xi|}$

Back to 2^{dim} reduce to a 1-dim integral via Green's Theorem

$$\int_{\partial D} A\,dx + B\,dy = \iint_D \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx\,dy$$



For us $D = B(0,1)$ $\partial D = S^1$.

Want: $\dfrac{\partial B}{\partial x} - \dfrac{\partial A}{\partial y} = e^{i(x,y)\xi}$ $\qquad \xi = (m,n)$

$A = e^{i\xi(x,y)} \dfrac{in}{|\xi|^2} = \dfrac{in}{m^2+n^2} e^{i(mx+ny)}$, $B = -e^{i\xi(x,y)} \dfrac{im}{|\xi|^2} = -\dfrac{im}{m^2+n^2} e^{i(mx+ny)}$

$$\left.\begin{aligned} \frac{\partial B}{\partial x} &= \frac{in(-im)}{m^2+n^2} e^{i(mx+ny)} \\ \frac{\partial A}{\partial y} &= \frac{in \cdot in}{m^2+n^2} e^{i(mx+ny)} \end{aligned}\right\} \Longrightarrow \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} = e^{i(mx+ny)}$$

$$\hat{\chi}(\xi) := \int_{|x|\le 1} e^{ix\cdot\xi} dx = \int_{S^1} \frac{in}{m^2+n^2} e^{i(mx+ny)} dx - \frac{im}{m^2+n^2} e^{i(mx+ny)} dy$$

Parametrize $\gamma(t) = (\cos t, \sin t) =$ unit speed parameterization of ∂D

$(dx, dy) = \dot{\gamma}(t)\, dt$

$$\hat{\chi}(\xi) = \int_0^{2\pi} \left\langle \left(\frac{in}{m^2+n^2}, \frac{-im}{m^2+n^2}\right), \dot{\gamma}(t) \right\rangle e^{i\langle \xi, \gamma(t)\rangle} dt =$$

$$= \frac{i}{|\xi|} \int_0^{2\pi} \left\langle \frac{\xi^\perp}{|\xi|}, \dot{\gamma}(t) \right\rangle e^{i\overbrace{|\xi|}^{\lambda} \overbrace{\langle \frac{\xi}{|\xi|}, \gamma(t)\rangle}^{\phi_\xi(t)}} dt$$

$\xi = (m,n)$, $\xi^\perp = (n, -m)$ $\quad \xi \cdot \xi^\perp = 0$

$$\hat{\chi}(\xi) = \frac{i}{|\xi|} \int_0^{2\pi} A_\xi(t) e^{i\lambda \phi_\xi(t)} dt , \quad \lambda = |\xi| \longrightarrow \infty$$

$A_\xi(t) = \left\langle \dfrac{\xi^\perp}{|\xi|}, \dot{\gamma}(t) \right\rangle$, $\phi_\xi(t) = \left\langle \dfrac{\xi}{|\xi|}, \gamma(t) \right\rangle$

Critical points of the phase function ϕ_ξ :

$\phi'_\xi(t) = \left\langle \dfrac{\xi}{|\xi|}, \dot{\gamma}(t) \right\rangle \overset{?}{=} 0 \iff \underset{(-\sin t, \cos t)}{\dot{\gamma}(t)} \perp \xi$

There will be 2 critical pts, t_0, $t_0 + \pi$.

$\gamma(t) = (\cos t, \sin t)$, $\dot{\gamma}(t) = (-\sin t, \cos t)$, $\ddot{\gamma}(t) = (-\cos t, -\sin t) = -\gamma(t)$

$$\phi''_\xi(t) = \left\langle \frac{\xi}{|\xi|}, \ddot{\gamma}(t) \right\rangle$$

$$\phi''(t) = 0 \iff \ddot{\gamma}(t) \perp \xi$$

Note We can not have both $\phi'(t) = 0 = \phi''(t) \iff \dot{\gamma}(t) \perp \xi, \ddot{\gamma}(t) \perp \xi$

$\Longrightarrow \ddot{\gamma}, \dot{\gamma}$ are parallel but $\dot{\gamma} \perp \ddot{\gamma}$. Real reason: γ is a unit speed param

$\dot{\gamma}(t)\dot{\gamma}(t) = 1 \Longrightarrow 0 = \frac{d}{dt} 1 = 2\ddot{\gamma}\cdot\dot{\gamma}$

VdC:

$$\left|\int_a^b A(t)e^{i\lambda\phi(t)}dt\right| \le \left\{ |A(b)| + \int_a^b |A'(t)|\,dt \right\} \frac{c_k}{(\min_{[a,b]} |\phi^{(k)}|)^{1/k}} \frac{1}{\lambda^{1/k}}$$

if $\phi^{(k)} \neq 0$ on $[a,b]$ (& monotone if $k=1$)

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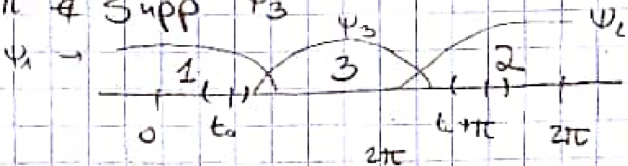
use a smooth partition of unity: $1_{[0,2\pi]} = \psi_1 + \psi_2 + \psi_3$

When ψ_j smooth, $\operatorname{supp}(\psi_1) \ni t_0$, $\operatorname{supp}(\psi_2) \ni t_0+\pi$
$\qquad\qquad\qquad\qquad \not\ni t_0+\pi \qquad\qquad \not\ni t_0$

& $t_0, t_0+\pi \notin \operatorname{supp} \psi_3$



$$\int A_\xi(t)\, e^{i\lambda\phi_\xi(t)} = \sum_{j=1}^{3} \int_0^{2\pi} \underbrace{A_\xi(t)\,\psi_j(t)}_{A_j(t)}\, e^{i\lambda\phi_\xi(t)}\,dt$$

For $j=1,2$ - use VdC 2, $j=3$ use VdC 1

Lets bound I_1, I_2 (each contains a single crit. pt)
us VdC 2

$\phi_\xi''(t) = \langle \frac{\xi}{|\xi|}, \ddot{\gamma}(t) \rangle$

Know $\ddot{\gamma}(t) \perp \dot{\gamma}(t)$, $\gamma(t_0) \perp \xi \implies \dot{\gamma}(t_0) = \pm \frac{\xi^\perp}{|\xi|}$

$\implies \ddot{\gamma}(t_0) = \pm \frac{\xi}{|\xi|}$

Near t_0, $\ddot{\gamma}(t)$ is close to $\xi/|\xi|$ (WLOG) so,

$\phi_\xi''(t) = \langle \frac{\xi}{|\xi|}, \ddot{\gamma}(t) \rangle \simeq \langle \frac{\xi}{|\xi|}, \frac{\xi}{|\xi|} \rangle = 1$

i.e $\phi_\xi''(t_0) = 1$ & is $\geq \frac{1}{2}$ say in $(t_0 - 1/10, t_0 + 1/10)$

$\implies |\phi''(t)| \geq 1/2$ if $\operatorname{supp} \psi_1 = [t_0 - 1/10, t_0 + 1/10]$

$|A_1(t)| = \underset{\overset{\wedge}{2}}{\psi_1(t)} \left| \langle \frac{\xi^\perp}{|\xi|}, \dot{\gamma}(t) \rangle \right| \le 2$, Likewise $A'(t) \le C$

C-S $\overset{\wedge}{} \left|\frac{\xi^\perp}{|\xi|}\right| \cdot |\dot{\gamma}(t)|$

$\implies I_1 \ll \frac{1}{|\xi|^{1/2}}$

Exercise: $I_3 \ll 1/|\xi|^{1}$

$$\Longrightarrow \quad \hat{\chi}(\xi) \ll \frac{1}{|\xi|^{3/2}}$$

General Thm:

Let $D \subset \mathbb{R}^2$ be a compact planar domain smooth bdry ∂D, s.t ∂D has nowhere zero curvature, then the Fourier transform of D:
$$\left|\iint_D e^{ix\xi}\,dx\right| \ll \frac{1}{|\xi|^{3/2}}$$

If $\gamma : [0,1] \longrightarrow \partial D$ is a unit parametrization $|\dot{\gamma}(t)| \equiv 1$, then the curvature is $\kappa(t) = |\ddot{\gamma}(t)|$

The formula of stationary phase
$$I(\lambda) = \int A(x)\, e^{i\lambda\phi(x)}\,dx$$

Assume $A \in C_c^\infty(\mathbb{R})$ smooth, $\phi \in C^\infty(\mathbb{R})$ real valued

Suppose ϕ has a single critical pt. $x_0 \in \mathrm{Supp}(A)$, $\phi'(x_0) = 0$, which is non-degenerate: $\phi''(x_0) \neq 0$.

Then
$$I(\lambda) \underset{\lambda \to \infty}{\sim} e^{\frac{i\pi}{4}\operatorname{sign}\phi''(x_0)} \cdot A(x_0)\sqrt{\frac{2\pi}{|\phi''(x_0)|}} \cdot \frac{e^{i\lambda\phi(x_0)}}{\sqrt{\lambda}}$$