26/3/2018

Lecture 4:

The circle problem:

$\mathbb{Z}^2 = \{(m,n) \mid m,n \in \mathbb{Z}\}$. Give a "nice" region $\Omega \subseteq \mathbb{R}^2$

$N_\Omega(R) = \# \mathbb{Z}^2 \cap R\Omega = \# \{(m,n) \in \mathbb{Z}^2 \mid (\frac{m}{R}, \frac{n}{R}) \in \Omega\}$

estimate $N(R)$ as $R \to \infty$.

Let Ω be an interval $\Omega = (\alpha, \beta) \subseteq \mathbb{R}$

$$N_\Omega(R) = \# \{n \in \mathbb{Z} \mid R\alpha < n < R\beta\}$$

Then: $N_\Omega(R) = \text{length}(R(\beta - \alpha)) + O(1) = R\,\text{length}(\Omega) + O(1)$.

For $\Omega =$ unit Disc, this is the circle problem.

First guess: $N_\Omega(R) \sim \text{area}(R\Omega) = R^2 \text{area}(\Omega)$

After we show this, then: the remainder:

$P(R) = N_\Omega(R) - \text{area}(R\Omega)$, the correct size of $P(R)$.

Easy: big O.

$$P(R) \underset{\uparrow\ \text{big } O}{\ll} \text{length}(\partial(R\Omega))$$

Will treat $\Omega = B(0,1)$ the circle/disk.

Proof 1:

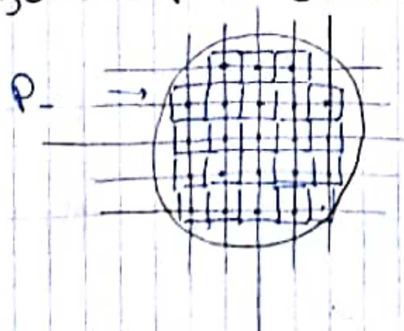
Let P_- = union of all unit squares centered at a lattice point $p \in \mathbb{Z}^2$ s.t $\Box_p \subset B(0,R)$. P_- is clearly a polygon, $P_- \subseteq R\Omega$

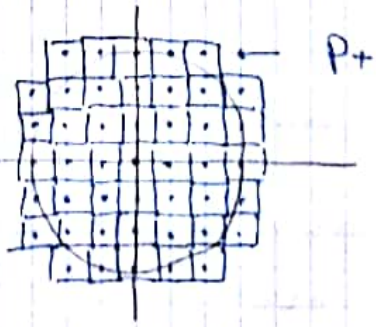
$$\# P_- \cap \mathbb{Z}^2 = \text{area } P_- = \sum_{p \in P_-} \underbrace{\text{Area}(\Box_p)}_{=1}$$

$P_+ = \underset{p \in \mathbb{Z}^2}{\text{union}} \Box_p$ s.t $\Box_p \cap B(0,R) \neq 0$. P_+ is a polygon, $B(0,R) \subseteq P_+$

$$\text{Area}(P_+) = \# P_+ \cap \mathbb{Z}^2$$

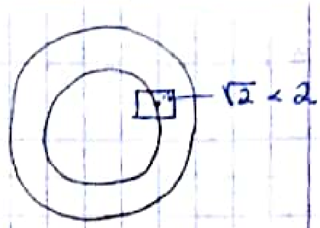
We got: $P_- \subseteq B(0,R) \subseteq P_+$





Claim:

$B(0, R-2) \subseteq P_-$ & $P_+ \subseteq B(0, R+2)$



Finally:

$$\text{Area } P_- = \# P_- \cap \mathbb{Z}^2 \le N(R) \le \# P_+ \cap \mathbb{Z}^2 = \text{area } P_+$$

We get: $\text{Area}(B(R-2)) \le N(R) \le \text{Area}(B(R+2))$

$$\pi(R-2)^2 = \pi R^2 - 4\pi R + 4 \qquad \pi(R+2)^2 = \pi R^2 + 4\pi R + 4$$

$$\Longrightarrow |N(R) - \pi R^2| \le 4\pi R + 4\pi = O(R)$$

So $N(R) = \pi R^2 + O(R)$

Proof:

Will work for any convex region with piecewise smooth boundary. (Think about exact requirments). Cleary we need boundary to have measure 0.

We found $N(R) = \text{Area}(R \cdot \Omega) + O(\text{length}(\partial R \cdot \Omega))$

"Circle problem":

$$N(R) - \text{area}(R \cdot \Omega) \ll (\text{length}(\partial(R \cdot \Omega)))^{\frac{1}{2} + \varepsilon} \qquad \forall \varepsilon > 0$$

Intuition: $N(R) - \text{area} = \sum \pm \Delta$

Assume that $\pm$ are "random" & the areas are on average $\frac{1}{2}$, & over/under estimations are independent. Then

$$\# \text{ of } \pm \Delta \approx \text{length(boundary)}$$

If we forget $\pm$ we get $O(R)$.

Central Limit Theorem:

$X_1, \dots, X_N$ IID random variables e.g. $X_i = \frac{1}{2}\varepsilon_i$

$$\varepsilon_i = \begin{cases} 1 & \text{prob } \frac{1}{2} \\ -1 & \text{prob } \frac{1}{2} \end{cases}$$

$S_N = X_1 + \dots + X_N - \frac{N}{2}$ will typicaly be of size $\sqrt{N}$.

$\Downarrow$

$\mathbb{E} X_i = 0 \quad S_N = \sum_{i=1}^{N} \{X_i - \underbrace{\mathbb{E}(X_i)}_{=0}\}, \quad \mathbb{E}(S_N) = 0$

$Var(S_N) = \mathbb{E}(S_N^2) = c \cdot N$

$Var(X_i) = \mathbb{E}(|X_i - \mathbb{E}(X_i)|^2) = \frac{1}{2} 1^{1/2} + \frac{1}{2} 1^{1/2} \quad 1^2 = \frac{1}{2}$

<u>CLT</u>:

$$\frac{S_N}{\sqrt{Var(S_N)}} \sim N(0,1) \quad Prob\left(a < \frac{S_N}{\sqrt{Var}} < b\right) \xrightarrow[N \to \infty]{} \int_a^b e^{-t^2/2} \frac{dt}{\sqrt{2\pi}} > 0$$

i.e. $\forall c > 0$, with positive prob. $\left|\frac{S_N}{\sqrt{N}}\right| > c$, but 100% of $\frac{S_N}{\sqrt{N}}$ is finite.

So we are led to expect that for most R's

$P(R) = N(R) - area(R\Omega)$ will be about $\sqrt{\text{length } \partial(R\Omega)}$

& we could expect $|P(R)|$ not to be much larger.

<u>Goal</u>:

1) Improve bound $P(R) = O(R)$

2) Can we give argument to show that we cannot do better then $O(R^{1/2})$

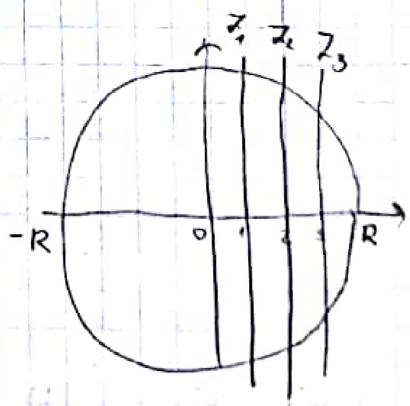
<u>Cramer 1920</u>:

$$\lim_{R \to \infty} \frac{1}{R} \int_0^R \left(\frac{P(r)}{\sqrt{r}}\right)^2 dr = c > 0$$

$\Longrightarrow$ for arbitrarily large R's we have $|P(R)| > * R^{1/2}$

i.e. "$P(R) = \Omega(R^{1/2})$"

<u>Proof 2</u>: (Slicing & reduce to 1-dim problem)



$$N(R) = \sum_{\substack{n \in \mathbb{Z} \\ |n| \le R}} \#(Z_n \cap \mathbb{Z}^2)$$

$Z_n = $ line (n,t), $t \in \mathbb{R}$, $n^2 + t^2 \le R^2$

$\# Z_n = \text{length}(Z_n) + O(1) \underset{?}{=} 2\sqrt{R^2 - n^2} + O(1)$

$n^2 + t^2 \le R^2 \Rightarrow t^2 \le R^2 - n^2$

By the one-dim estimate.

$$N(R) = \sum_{|n| \le R} \{2\sqrt{R^2 - n^2} + O(1)\} = 2 \cdot 2 \sum_{n=1}^{R} \sqrt{R^2 - n^2} + 2R + O(1) + O(R) =$$

$$= 4 \sum_{n=1}^{R} \sqrt{R^2 - n^2} + O(R)$$

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$$\sum_{n=1}^{R} \sqrt{R^2-n^2}$$

use summation by parts $\sum_{0<n\le R} a_n f(n) = A(R)f(R) - \int_0^R A(t)f'(t)\,dt$, $A(t) = \sum_{0<n\le t} a_n$

take $f(t) = R^2-t^2$, $a_n = 1$, $A(t) = \sum_{0<n\le t} 1 = \lfloor t \rfloor$, $f' = \frac{-t}{\sqrt{R^2-t^2}}$

$$\sum_{1\le n\le R} \sqrt{R^2-n^2} = \underbrace{\lfloor R \rfloor \sqrt{R^2-R^2}}_{=0} - \int_0^R \lfloor t \rfloor \frac{-t}{\sqrt{R^2-t^2}}\,dt =$$

$$= \int_0^R \frac{(t-\{t\})\,t}{\sqrt{R^2-t^2}}\,dt = \int_0^R \frac{t^2}{\sqrt{R^2-t^2}}\,dt + O\Big(\underbrace{\int_0^R \frac{t}{\sqrt{R^2-t^2}}\,dt}_{= -\sqrt{R^2-t^2}\Big|_0^R = R}\Big)$$

$$\int_0^R \frac{t^2}{\sqrt{R^2-t^2}}\,dt = \int_0^1 \frac{R^2u^2}{R(1-u^2)}\,R\,du = R^2 \underbrace{\int_0^1 \frac{u^2}{\sqrt{1-u^2}}\,du}_{\pi/4}$$

$t = Ru$
$dt = R\,du$

We find:

$$N(R) = 4\sum_{n=1}^{\lfloor R \rfloor} \sqrt{R^2-n^2} + O(R) = \pi R^2 + O(R)$$

Dirichlet's divisor problem (average number of divisors of an integer)

$$d(n) = \#\text{ positive divisors of } n = \sum_{\substack{d\mid n\\ d>0}} 1 = \#\{(x,\eta) \mid x,\eta \ge 1,\ x\eta = n\}$$
integers

formula: (in terms of prime decomposition of n)

$d(1) = 1$, $d(2) = 2$, $d(3) = 2$, $d(4) = 3$

$d(p) = 2 \iff p$ prime, $d(p^k) = k+1$
$1, p, p^2, \dots, p^k$

Multiplicativity:

If m, n coprime $\gcd(m,n) = 1$ then $d(mn) = d(m)\,d(n)$

$\implies n = p_1^{k_1} \cdots p_r^{k_r}$, p_i distinct primes, $d(n) = \prod_{j=1}^{r} (k_j+1)$

Question: $\frac{1}{N} \underbrace{\sum_{n=1}^{N} d(n)}_{D(N)} = \frac{D(N)}{N}$ claim $\frac{D(N)}{N} \underset{N\to\infty}{\sim} \log N$

"The average number of divisors of n is $\log n$"

Note:

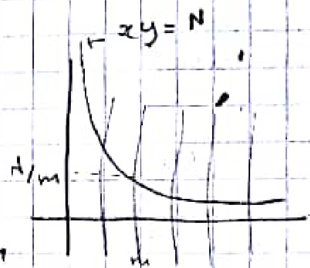
$$D(N) = \sum_{n=1}^{N} \underset{\#\{(x,y)\in\mathbb{Z}^2 \mid x,y\geq 1,\ x\cdot y = n\}}{\underset{\parallel}{d(n)}} = \#\{(x,y)\in\mathbb{Z}^2 \mid x,y\geq 1,\ x\cdot y\leq N\}$$

Thm. (Dirichlet)

$$D(N) = N\log N + (2c-1)N + \Delta(N), \quad \text{where } \Delta(N) = O(\sqrt{N})$$

Dirichlet divisor problem:

$$\forall \varepsilon > 0. \quad \Delta(N) = O(N^{1/4+\varepsilon})$$



First attempt: $D(N) \sim N\log N$

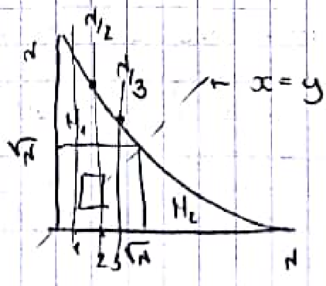
Use slicing method

$$D(N) = \sum_{1\leq m\leq N} \lfloor N/m \rfloor = \sum_{1\leq m\leq N} \left\{\frac{N}{m} + O(1)\right\} =$$

$$= N\sum_{m=1}^{\lfloor N\rfloor} \frac{1}{m} + O(N) = N\left(\log(N) + c + O\left(\tfrac{1}{N}\right)\right) = O(N) = N\log N + O(N).$$

$$\left(\sum_{m=1}^{x} \frac{1}{m} = \log x + c + O\left(\frac{1}{x}\right)\right)$$

Second attempt



The "hyperbola method"

$$D(N) = \square + H_1 + H_2 = \square + 2H_1, \quad H_1 = H_2$$

$$\square = \#\{(x,y)\in\mathbb{Z}^2 \mid 1\leq x,y\leq\sqrt{N}\} = \left(\#\{1\leq x\leq\sqrt{N}\}\right)^2$$

$$= (\sqrt{N} + O(1))^2 = N + O(\sqrt{N})$$

For H_1 use the same slicing method.

$$H_1 = \sum_{1\leq m\leq\sqrt{N}} \#\left\{\sqrt{N} < k \leq \frac{N}{m}\right\} = \sum_{1\leq m\leq\sqrt{N}} \left\{\left(\frac{N}{m} - \sqrt{N}\right) + O(1)\right\} =$$

$$= N\sum_{m=1}^{\lfloor\sqrt{N}\rfloor} \frac{1}{m} - \sqrt{N}\lfloor\sqrt{N}\rfloor + O(\sqrt{N}) = N\left\{\overbrace{\log\sqrt{N}}^{\frac{1}{2}\log N} + c + O\left(\tfrac{1}{\sqrt{N}}\right)\right\} - N + O(\sqrt{N})$$

$$H_1 = \frac{1}{2}N\log N + cN - N + O(\sqrt{N})$$

$$D(N) = \square + 2H_1 = N + O(\sqrt{N}) + 2\left\{ \tfrac{1}{2} N \log N + CN - N + O(\sqrt{N}) \right\} =$$

$$= N \log N + (2C-1)N + O(\sqrt{N})$$

<u>What we will do next time:</u>

<u>1906 Sierpinski, ...,</u> : $N(R) = \pi R^2 + O(R^{2/3})$

$$\tfrac{1}{2} < \tfrac{2}{3} < 1 \qquad \left(\tfrac{2}{3} = 0.666\ldots \text{, best known result today: } 0.6\ast\ldots \right)$$

<u>Vander Corput ~1920</u> : replaced $\frac{2}{3}$ by $\frac{2}{3} - \delta$.

Likewise for $\Delta(R)$.

<u>Recall</u>:

We argued that $N(R) - \pi R^2$ is $O(R^{1/2+\varepsilon})$

by "using" independence of errors in approximation,

$$P(R) = N(R) - \pi R^2 = \sum \pm \Delta$$

$$? \Longrightarrow \frac{P(R)}{\sqrt{R}} \text{ is gaussian ??}$$

$$\frac{1}{R} \operatorname{meas}\left\{ u < r < R \;\middle|\; \alpha < \frac{P(r)}{\sqrt{R}} < \beta \right\} \xrightarrow[R \to \infty]{??} \int_\alpha^\beta e^{-x^2} dx$$

<u>1940's Wintner</u> showed $\exists$ limit

<u>1990 : Heath Brown</u> : distribution is NON GAUSSIAN.

<u>For next time</u>: 1) read up on basic Fourier transform.

Poission Summation, Fourier inversion, ...

2) oscillatory integrals, the method of statinary phase.