Lecture 2: 14/3/2018

Reminder:

Let $X = \{x_n\}$ be a sequence of reals. We say X is uniformly distributed mod 1 if $\forall$ every subinterval $I \subset [0,1)$

$$\frac{1}{N} \# \{ \{x_n\} \in I \mid n \le N \} \xrightarrow[N\to\infty]{} \text{length}(I)$$

$\{x_n\}$ ↑ fractional part

$\Longrightarrow$ The fractional parts $\{x_n\}$ are dense in $[0,1)$.

The converse is false. $x_n = \log n$ then we saw that they are not u.d. exercise $\{\log n\}$ are dense in $[0,1)$.

Weyl's Criterion:

$\forall\, 0 \ne k \in \mathbb{Z}$ $\quad \frac{1}{N}\sum_{n=1}^{N} e(kx_n) \longrightarrow 0 \qquad e(z) := e^{2\pi i z}$

$\Longleftrightarrow$ Fractional part of x_n are u.d.

Example:

$x_n = \alpha n$, $\alpha \notin \mathbb{Q} \Longrightarrow$ u.D mod 1

Proof:

Show $\sum_{n=1}^{N} e(k\alpha n) = o(N)$ $\forall k \ne 0$.

Indeed: $\sum_{n=1}^{N} e(k\alpha_n) = \sum_{n=1}^{N} e(k\alpha)^n = \dfrac{e(k\alpha) - e(k\alpha(N+1))}{1 - e(k\alpha)}$

$\underset{1}{x} + \underset{1}{x}^2 + \dots + x^N = \dfrac{x - x^{N+1}}{1-x}$ & $\alpha \notin \mathbb{Q}$ so $e^{2\pi i k\alpha} \ne 1$ $\forall k \ne 0$.

$$\left|\sum_{n=1}^{N} e(k\alpha n)\right| \le \frac{|e(k\alpha)| + |e(k\alpha(N+1))|}{|1 - e(k\alpha)|} \qquad \left(|e^{it}| = 1 \quad \forall t \in \mathbb{R}\right)$$

$$1 - e(k\alpha) = 1 - e^{2\pi i k\alpha} = e^{\pi i k\alpha}\underbrace{\left(e^{-i\pi k\alpha} - e^{i\pi k\alpha}\right)}_{= -2i\sin(\pi k\alpha)}$$

So $\left|\frac{1}{N}\sum_{n=1}^{N} e(k\alpha n)\right| < \frac{1}{N}\,\text{const}(k\alpha) \xrightarrow[N\to\infty]{} 0$

How to prove Weyl's Criterion:

Observe U.D $\iff$ $\underbrace{\frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_I(x_n)}_{\#\{n \le N \mid x_n \in I\}} \longrightarrow \underbrace{\int_0^1 \mathbb{1}_I(x)\,dx}_{=\text{length}(I)}$

$$\mathbb{1}_I(x) := \begin{cases} 1 & x \in I \\ 0 & \text{else} \end{cases} \quad \forall\, I \subset [0,1) \text{ interval.}$$

Lemma: $(*)$ $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\,dx$

The following are equivalent:

1) $\{x_n\}$ is U.D. $(*)$ holds $\forall f = \mathbb{1}_I$

obvious. 2) $(*)$ holds for all continuous f on $[0,1)$

3) $(*)$ holds for all Riemann-integrable f.

1) $\Rightarrow$ 3):

Recall that f is Riemann-integrable iff $\forall \varepsilon > 0$ $\exists$ step function (finite linear comb. of indicator function of intervals) S_+, S_- s.t.

1) $S_- \le f \le S_+$

2) $\int (S_+ - S_-) < \varepsilon$

Now UD $\iff$ $(*)$ for indicator functions $\Rightarrow$ $(*)$ for step function

$\Big(s = \mathbb{1}_I + 2\mathbb{1}_J \quad \frac{1}{N}\sum_{n=1}^{N} s(x_n) = \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_I(x_n) + 2\frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_J(x_n) \longrightarrow$

$\qquad\qquad \int \mathbb{1}_I \qquad\qquad \int \mathbb{1}_J$

$\longrightarrow \int (\mathbb{1}_I + 2\mathbb{1}_J) = \int s(x)\,dx \Big)$

$$\frac{1}{N}\sum f(x_n) - \int f(x) = \underbrace{\frac{1}{N}\sum_{n=1}^{N}\big(f(x_n) - S_+(x_n)\big)}_{\le 0} + \frac{1}{N}\sum_{n=1}^{N} S_+(x_n)$$

$$- \underbrace{\int_0^1 S_+(x)\,dx}_{\mathrm{I}} + \underbrace{\int_0^1 (S_+(x) - f(x))\,dx}_{\mathrm{II}} \le$$

$\le \mathrm{I} + \mathrm{II}$

Show: $\exists\, N(\varepsilon)$ s.t. $\forall N \ge N(\varepsilon)$, $|\mathrm{I}| < \varepsilon$ – ok, because by (1), $(*)$ holds for step functions like S_+.

$$\mathrm{II} = \int_0^1 (S_+ - f) \le \int_0^1 (S_+ - S_-) < \varepsilon$$

<u>Hence:</u> $\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_0^1 f(x) < 2\varepsilon$

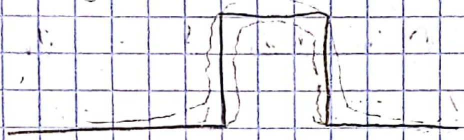
Likewise replace S_+ by S_- will get $\frac{1}{N}\sum f(x_n) - \int f > -2\varepsilon$

$\Longrightarrow \forall \varepsilon > 0, \exists N(\varepsilon)$ s.t $\forall N \ge N(\varepsilon)$

$$\left| \frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_0^1 f(x)\,dx \right| < 2\varepsilon$$ ie (*) holds for f.

<u>2) $\Longrightarrow$ 1):</u>

Approximate indicator functions by continuous as before:



<u>Weyl's Criterion:</u>

(*) holds for all $k \in \mathbb{Z}$

$$\frac{1}{N}\sum_{n=1}^{N} e(kx_n) \longrightarrow \int_0^1 \underset{e^{2\pi i k x}}{\underset{\shortparallel}{e_k(x)}}\,dx = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

By linearity, Weyl's Criterion $\iff$ (*) for all trigonometric pol.s.

$$T(x) = \sum_{k=1}^{K} c_k e^{2\pi i k x}$$

<u>Claim:</u>

(*) holds $\forall$ trig. pols $\iff$ (*) holds $\forall$ continuous fn.s $\overset{\text{Lemma}}{\iff}$ uD

<u>Weierstass' Approximation thm.:</u>

Every continuous fn. on $[0,1)$ is <u>uniformly</u> approx by trig pols.

$\forall \varepsilon > 0, \exists T_\varepsilon$ trig. pol s.t $\|f - T_\varepsilon\|_\infty < \varepsilon$

$$\|f - T_\varepsilon\|_\infty = \sup_{x\in[0,1]} |f(x) - T_\varepsilon(x)|$$

So take f cont, $\varepsilon > 0$, T_ε as above

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_0^1 f(x)\,dx = \underbrace{\frac{1}{N}\sum_{n=1}^{N} \big(f(x_n) - T_\varepsilon(x_n)\big)}_{\mathrm{III}} +$$

$$+ \underbrace{\frac{1}{N}\sum T_\varepsilon(x_n) - \int_0^1 T_\varepsilon(x)}_{\mathrm{I}} + \underbrace{\int_0^1 \big(T_\varepsilon(x) - f(x)\big)\,dx}_{\mathrm{II}}$$

Take $N(\varepsilon)$ s.t $\forall N \geq N(\varepsilon)$, $|I| < \varepsilon$ (ok because assuming (*) holds for trig poly's)

$$|II| \leq \int_0^1 |T_\varepsilon(x) - f(x)|\, dx < \varepsilon$$

$$|III| \leq \frac{1}{N} \sum_{n=1}^{N} \underbrace{|f(x_n) - T_\varepsilon(x_n)|}_{<\varepsilon} < \varepsilon$$

$$\Longrightarrow \left| \frac{1}{N} \sum f(x_n) - \int f \right| < 3\varepsilon \quad \forall N \geq N(\varepsilon)$$

$\blacksquare$

Thm. (Weyl, 1916)

The sequence αn^2 is U.D mod 1 if $\alpha \notin \mathbb{Q}$

Start of Proof:

Need to show:

$$\forall k \geq 0 \qquad S(N) = \sum_{n=1}^{N} e(\alpha k n^2) = o(N)$$

Weyl's differencing trick:

$$|S(N)|^2 = \sum_{n=1}^{N} e(\alpha k n^2) \overline{\sum_{m=1}^{N} e(\alpha k m^2)} = , \quad \overline{e(z)} = e(-z) \text{ if } z \in \mathbb{R}$$

$$= \sum_{n=1}^{N} e(\alpha k n^2) \sum_{m=1}^{N} e(-\alpha k m^2) = \sum_{n=1}^{N} \sum_{m=1}^{N} e(\alpha k (n^2 - m^2)) =$$

$$= \sum_{m=n=1}^{N} 1 + \sum\sum_{m \neq n} e(\alpha k (n^2 - m^2)) = \quad e(z) + e(-z) = 2\mathrm{Re}(e(z))$$

$$= N + 2\mathrm{Re} \sum_{1 \leq m \lneq n \leq N} e(\alpha k (n^2 - m^2)).$$

Write $n = m + h$, $1 \leq h \leq N - m$, $1 \leq m \leq N - 1$

or $1 \leq h \leq N - 1$, $1 \leq m \leq N - h$.

$$n^2 - m^2 = (n-m)(n+m) = h(2m+h) = h^2 + 2hm$$

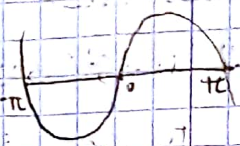
$$\text{off} := \sum\sum_{1 \leq m < n \leq N} e(\alpha k \underbrace{(n^2 - m^2)}_{h^2 + 2hm}) = \sum_{h=1}^{N-1} e(\alpha k h^2) \underbrace{\sum_{m=1}^{N-h} e((\alpha k 2h) \cdot m)}_{\text{a geometric progression}}$$

$$|\text{off}| \leq \sum_{h=1}^{N-1} \left| \sum_{m=1}^{N-h} e((\alpha k 2h) m) \right|$$

α irrational $\Rightarrow$ this is never 1.

want $o(N^2)$

$$\left|\sum_{m=1}^{N-h} e(\underbrace{2\alpha k}_{\gamma} \cdot h \cdot m)\right| \le \frac{1}{|\sin(\pi\gamma h)|}$$



$$|\text{Off}| \le \sum_{k=1}^{N-1} \min\left(\frac{1}{|\sin(\pi\gamma h)|}, N\right)$$

$|\sin(y)| \ge \frac{1}{2}|y|$

$|y| \le \frac{1}{10}$

Proposition:

$\gamma \notin \mathbb{Q}$ then $\sum_{h=1}^{N-1} \min\left(N, \frac{1}{\|\gamma h\|}\right) = o(N^2)$ where $\|x\| := \text{dist}(x, \mathbb{Z})$

(This will prove Weyl's Criterion)

Idea: replace γ by a rational approximation

Dirichlet's lemma:

Given $x \in \mathbb{R}$, $N \ge 1$ $\exists\, a \in \mathbb{Z}$ $1 \le q \le N$ s.t. $\left|x - \frac{a}{q}\right| < \frac{1}{qN}$.

Moreover, if $x \notin \mathbb{Q}$ then $q \to \infty$ as $N \to \infty$.

Proof:

Divide $[0,1)$ into N subintervals of length $\frac{1}{N}$ (0, $\frac{1}{N}$, $\frac{2}{N}$, 1)

consider the $N+1$ pts $0, \{x\}, \{2x\}, \{3x\}, \dots, \{Nx\} \in [0,1)$.

$\Longrightarrow \exists$ subinterval which contains $\{mx\}, \{nx\}$, $0 \le m \ne n \le N$

$\{y\} := y - \underbrace{\lfloor y \rfloor}_{\text{integer}}$

$$0 \le \{mx\} - \{nx\} < 1/N$$

$$\{mx\} = mx - \underbrace{\lfloor mx \rfloor}_{M_1}, \quad \{nx\} = nx - \underbrace{\lfloor nx \rfloor}_{M_2} \qquad M_1, M_2 \in \mathbb{Z}$$

$$0 \le \underbrace{(m-n)}_{q} x - \overbrace{(M_1 - M_2)}^{a} < \frac{1}{N}$$

Say $q := m - n > 0$ then $1 \le q \le N$, set $a := M_1 - M_2 \in \mathbb{Z}$

$$0 \le qx - a < \frac{1}{N} \longrightarrow 0 < x - \frac{a}{q} < \frac{1}{qN}$$

Moreover, If $x \notin \mathbb{Q}$ then $q \to \infty$ with $N \to \infty$

else, $\exists$ infinitly N's st (say) $0 < \left|x - \frac{3}{7}\right| < \frac{1}{7N} \xrightarrow[N \to \infty]{} 0$

Remarks:

1) Proof is not constructive. Can use continued fractions to get such approximations.

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 \ldots}}} \qquad a_0 \in \mathbb{Z},\ a_1, a_2, a_3, \ldots \geq 1$$

take n'th convergent $\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \frac{1}{a_n}}}$ then:

$$\left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}$$

Examples:

→ $x = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 \ldots}}$ $\frac{p_n}{q_n} = \frac{F_{n+1}}{F_n}$, F_n = n'th Fibonacci 0, 1, 1, 2, 3, 5, 8, 13, 21, ...

→ x = quadratic irrationality

$\frac{a + \sqrt{D}}{b} \iff$ periodic cont. frac. expansion

$D, a, b \in \mathbb{Z}$.

Quadratic irrationalities are BADLY approximable by rationals:

$$\exists\, c = c(x).\ \left|x - \frac{p}{q}\right| \geq \frac{c}{q^2} \quad \forall p, q.$$

Example:

→ $x = 0.\overset{1}{1}\overset{2}{1}000\overset{6=3!}{1}0 \ldots 0\overset{24=4!}{1}0 \ldots 0\overset{120=5!}{1}0 \ldots$

(Liouville) is extremely well approximable:

$$\left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{n+1}} \qquad q_n = 10^{n!}$$

Liouville:

If x is real algebraic of degree $d \geq 1$ then $\exists\, c(x) > 0$ s.t. if $x \neq \frac{p}{q}$ then $\left|x - \frac{p}{q}\right| \geq \frac{c}{q^d}$ $\Rightarrow$ x is Transcendental.

Thue, Siegel, Dyson, ..., Roth (1950):

x is algebraic irrational, $\forall \varepsilon > 0, \exists\, c(x,\varepsilon) > 0$ s.t. $\left|x - \frac{p}{q}\right| \geq \frac{c}{q^{2+\varepsilon}}$

Application:

Show finiteness of # of solutions of Diophantine eqs.

Bad example: Pell's equation $x^2 - 2y^2 = 1$ has ∞ many integer sols. Start from $(3,2)$ $(3+2\sqrt{2})^n = x_n + \sqrt{2}y_n$ then $x_n^2 - 2y_n^2 = 1$

Thue's equation:

$$x^3 - 2y^3 = 1$$

Axel Thue 1909 only finitely many integer sols.

Roth's Thm: $\Rightarrow \left|\sqrt[3]{2} - \frac{p}{q}\right| > \frac{c}{q^{5/2}}$ $\quad\left(\frac{5}{2} < 3\right)$

$$\frac{1}{|y|^3} = \left|\left(\frac{x}{y}\right)^3 - 2\right| = \left|\frac{x}{y} - \sqrt[3]{2}\right|\left|\frac{x}{y} - \sqrt[3]{2}e^{2\pi i/3}\right|^2 > \frac{c}{y^{5/2}}\left(\frac{\sqrt{3}}{2}\right)^2$$

$$z^3 - 2 = (z - \sqrt[3]{2})(z - \sqrt[3]{2}e^{2\pi i/3})(z - \sqrt[3]{2}e^{-2\pi i/3})$$

$$\frac{1}{y^3} > \frac{c}{y^{5/2}} \iff y^{1/2} < \frac{1}{c} \qquad |y| < c''$$

So x, y have only finitely many solutions.

(Alan Baker "linear forms in logarithms")

Prop:

$\gamma \notin \mathbb{Q}$ then $\displaystyle\sum_{h=1}^{N} \min\left(N, \frac{1}{\|h\gamma\|}\right) = o(N^2)$

$\left(\Rightarrow \{\alpha n^2\} \text{ is UD mod } 1, \forall \alpha \notin \mathbb{Q}\right)$

Proof:

Use Dirichlet's lemma to find coprime a, q, $1 \leq q \leq N$, $\left|\gamma - \frac{a}{q}\right| < \frac{1}{qN}$

i.e. $\gamma = \frac{a}{q} + \frac{\theta}{qN}$, $|\theta| < 1$

$\|h\gamma\| = \left\|\frac{ha}{q} + \frac{\theta h}{qN}\right\|$

$\quad\parallel$

$\operatorname{dist}(h\theta, \mathbb{Z})$

Note: $h \longrightarrow h + kq$ then you haven't changed this expression by much.

Write $(M-1)q < N \le M \cdot q$

Our sum $\le \sum_{h=1}^{Mq} \min\left(N, \frac{1}{\|h\gamma\|}\right) \le \sum_{k=1}^{M-1} \max_{|\tilde{\theta}|<1} \sum_{j=1}^{q} \min\left(N, \frac{1}{\|\frac{a(kq+j)}{q} + \frac{\tilde{\theta}}{q}\|}\right)$

$h = kq + j$

$j = 1, 2, \ldots, q$

$0 \le k < M \approx N/q$

$$= M \cdot \max_{|\tilde{\theta}|<1} \sum_{j=1}^{q} \min\left(N, \frac{1}{\|\frac{aj}{q} + \frac{\tilde{\theta}}{q}\|}\right) =$$

change variables: As j runs over $\mathbb{Z}/q\mathbb{Z}$, so does aj

since $(a,q) = 1$. $aj \mapsto i$

$$= M \cdot \max_{|\tilde{\theta}|<1} \sum_{i=1}^{q} \min\left(N, \frac{1}{\|\frac{i}{q} + \frac{\tilde{\theta}}{q}\|}\right)$$

There is at most ONE value of i such that $\|\frac{i+\tilde{\theta}}{q}\| < \frac{1}{2q}$

for this i, take N in minimum. For other i's

$$\left\|\frac{i}{q} + \frac{\tilde{\theta}}{q}\right\| > \frac{1}{2}\min\left(\frac{i}{q}, \frac{q-i}{q}\right)$$

So our sum $\ll \frac{N}{q}\left(N + 2\sum_{i=1}^{q/2} \frac{1}{i/q}\right) \ll$

$$\ll \frac{N^2}{q} + \frac{N}{q} q \cdot \underbrace{\sum_{i=1}^{q} \frac{1}{i}}_{\ll \log q \le \log N} \ll \frac{N^2}{q} + \underbrace{N \log N}_{o(N^2)}$$

Now recall $q \to \infty$ as $N \to \infty$ since $\gamma \notin \mathbb{Q}$ so $\frac{N^2}{q} = o(N^2)$

<u>Thm. (Weyl, 1916)</u>:

$P(x) = a_d x^d + a_{d-1}x^{d-1} + \ldots + a_1 x + a_0$ polynomial s.t at least one of $a_d, a_{d-1}, \ldots, a_1$ (not including the a_0) is irrational. Then the sequence $\{P(n) \mid n = 1, 2, \ldots\}$ is UD mod 1.

See book Kuipers & Niederreiter for streamlined argument due to van der Corput.