Lecture 10: 23/5/2018

Applications to eigenfunctions of the Laplacian:

$\Omega \subseteq \mathbb{R}^2$ nice planar domain with piecewise smooth boundary $\partial\Omega$. $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

acting on functions vanishing near $\partial\Omega$.

Self adjoint operator Δ:

$$\int_\Omega f \Delta g = \int_\Omega \Delta f g \quad \text{- "Dirichle Laplacian"}$$

Fact:

$L^2(\Omega)$ has ONB of eigenfunctions of Δ.

Example: $\Delta = \frac{d^2}{dx^2}$ on $L^2(\mathbb{R})$. Self adjoint $\int f g'' = \int f'' g$.

No eigenfunctions in $L^2(\mathbb{R})$

$$\Delta f = f'' = -k^2 f \Rightarrow f = a_+ e^{ikx} + a_- e^{-ikx} \notin L^2$$

$$k \in \mathbb{R},\ -k^2 < 0$$

$$L^2(\mathbb{R}) \ni f(x) = \int \hat{f} e^{ikx} dk$$

Example 1:

$\dim 1$, $\Omega = [0, A]$, $f_n(x) = \sqrt{\frac{2}{A}} \sin\left(\pi n \frac{x}{A}\right)$, $n \geq 1$

$f_n''(x) = -\left(\frac{\pi n}{A}\right)^2 f_n(x)$, $f_n|_{\partial\Omega} = 0$. ONB of $f \in L^2$ vanish on $\partial\Omega$.

Weyl's Law (1911):

For such $\Omega \subseteq \mathbb{R}^2$ $\quad \#\{E_n \leq x\} \sim \frac{\text{area}(\Omega)}{4\pi} x$, $x \to \infty$.

e.g. $\Omega = A \,\boxed{}\, B$ is $f_{m,n} = \frac{2}{\sqrt{AB}} \sin\left(\pi m \frac{x}{A}\right) \sin\left(\pi n \frac{y}{B}\right)$

eigenvalues $E_{m,n} = \left(\frac{\pi m}{A}\right)^2 + \left(\frac{\pi n}{B}\right)^2$. $m, n \geq 1$.

$$\left(\frac{\pi x}{A}\right)^2 + \left(\frac{\pi y}{B}\right)^2 \leq E \leftarrow \text{count solutions}$$

"hearing the area of Ω"

Thm. (Kac, McKean & Singer)

If Ω has smooth boundary then

$$\underbrace{\sum_{n=1} e^{-tE_n}}_{\text{trace of the heat kernel}} \underset{t\searrow 0}{\sim} C_1 \frac{\text{area}(\Omega)}{\sqrt{t}} - C_2 \frac{\text{length}(\partial\Omega)}{\sqrt{t}} + \frac{1-h}{6} + o(1)$$

h = #holes

In general: M is a d-dim compact smooth Riemann manifold, then $-\Delta_g f_n = E_n f_n$.

$$\sum_{n\geq 0} e^{-tE_n} = \frac{1}{(4\pi t)^{n/2}} \left\{ \text{Vol}(M) + a_1 t + a_2 t^2 + \dots \right\}$$

dim $d = 2$

$$\sum{}' e^{-tE_n} \underset{t\searrow 0}{\sim} \frac{1}{4\pi t} \left\{ \text{area}(M) + \frac{1-g(M)}{12} + O(t^2) \right\}$$

$g(M)$ = genus of M.

Let's compute the small time asymptotics of $K(t) = \sum{}' e^{-tE_n}$ for a rectangle Ω_{AB} = 

Method: reduce to 1-dim:

Eigenvalues $E_{m,n} = \left(\frac{\pi m}{A}\right)^2 + \left(\frac{\pi n}{B}\right)^2$

$$K(t) = \sum{}' e^{-tE_{m,n}} = \sum_{m\geq 1}{}' e^{-t(\pi m/A)^2} \sum_{n\geq 1}{}' e^{-t(\pi n/B)^2} = K_{\Omega_A}(t) K_{\Omega_B}(t)$$

$\Omega_A = [0, A]$ interval.

And we get $\sum\sum{}' e^{-\tau(n^2+m^2)} = \left(\sum{}' e^{-\tau n^2}\right)\left(\sum{}' e^{-\tau m^2}\right)$

1-dim $K_{[0,A]}(t) = \sum_{n=1}^{\infty}{}' e^{-t\left(\frac{\pi n}{A}\right)^2} = \dfrac{\sum_{n=-\infty}^{\infty}{}' e^{-\tau\pi n^2} - 1}{2}$ $\ominus$ $\left(\tau = \frac{\pi}{A^2} t \right)$

When $\Theta(\tau) = \sum_{n\in\mathbb{Z}} e^{-\pi\tau n^2}$ Theta function

$\ominus$ $\dfrac{\Theta(\tau) - 1}{2}$

Note: as $\tau \to \infty$ $\Theta(\tau) = 1 + O(e^{-\pi\tau})$

$$\sum_{n=1}^{\infty} e^{-\tau n^2} \leq \sum_{k=1}^{\infty} e^{-\tau k} = \frac{e^{-\tau}}{1-e^{-\tau}}$$

Thm.:

$$\Theta\left(\frac{1}{\tau}\right) = \sqrt{\tau}\,\theta(\tau)$$

Cor.:

$$\Theta(\tau) \sim \frac{1}{\sqrt{\tau}} + O(e^{-\pi/\tau}), \quad \tau \searrow 0$$

$$\Theta(\tau) = \frac{1}{\sqrt{\tau}}\,\Theta\left(\frac{1}{\tau}\right) \underset{\substack{\tau \searrow 0 \\ 1/\tau \nearrow \infty}}{=} \frac{1}{\sqrt{\tau}}\left(1 + O(e^{-\pi/\tau})\right)$$

Cor.:

The 1-dim heat kernel

$$K_{[0,A]}(t) = \frac{\Theta(t)-1}{2} = -\frac{1}{2} + \frac{1}{2}\left\{\frac{1}{\sqrt{\frac{\pi}{A^2}t}} + \text{small}\right\}$$

$$K_{[0,A]}(t) = \frac{A}{2\sqrt{\pi}}\cdot\frac{1}{\sqrt{t}} - \frac{1}{2} + \text{small}$$

For rectangle:

$$K_{\Omega_{A,B}}(t) = K_{\Omega_A}(t)\cdot K_{\Omega_B}(t) = \left(\frac{A}{2\sqrt{\pi}}\cdot\frac{1}{\sqrt{t}} - \frac{1}{2} + \dots\right)\left(\frac{B}{2\sqrt{\pi}}\cdot\frac{1}{\sqrt{t}} - \frac{1}{2} + \dots\right) =$$

$$= \frac{AB}{4\pi}\cdot\frac{1}{t} - \frac{A+B}{4\sqrt{\pi}}\cdot\frac{1}{\sqrt{t}} + \frac{1}{4} + o(1) =$$

$$= \frac{\text{Area}(\Omega)}{4\pi}\cdot\frac{1}{t} - \frac{\text{length}(\partial\Omega)}{8\sqrt{\pi}}\cdot\frac{1}{\sqrt{t}} + \frac{1}{4} + o(1)$$

Compare: In general smooth $\Omega \subseteq \mathbb{R}^2$,

$$K_\Omega(t) = \frac{\text{area}(\Omega)}{4\pi}\cdot\frac{1}{t} - \frac{1}{8\sqrt{\pi}}\,\frac{\text{length}}{\sqrt{t}} + o(1)$$

Notice: A rectangle is not smooth.

Lets prove that $\Theta(1/\tau) = \sqrt{\tau}\,\Theta(\tau)$

proof:

$\Theta(\tau) = \sum_{n\in\mathbb{Z}} e^{-\pi\tau n^2}$. Let $g(x) = e^{-\pi x^2}$.

$\tau > 0$, abs. Convergent. $g_\tau(x) = g(\sqrt{\tau}\,x) = e^{-\pi\tau x^2}$

$$\Theta(\tau) = \sum_{n\in\mathbb{Z}} g_\tau(n) \underset{\substack{\uparrow \\ \text{Poisson Summation}}}{=} \sum_{m\in\mathbb{Z}} \widehat{g_\tau}(m)$$

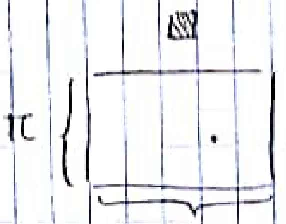
Recall $\forall y$. $g_\lambda(y) = g(\lambda y)$ then $\hat{g}_\lambda(y) = \frac{1}{\lambda}\hat{g}\left(\frac{y}{\lambda}\right)$.

Also for $g(x) = e^{-\pi x^2}$, $\hat{g} = g \Rightarrow$

$$\widehat{g_{\sqrt{\tau}}}(x) = \frac{1}{\sqrt{\tau}}\hat{g}\left(\frac{x}{\sqrt{\tau}}\right) = \frac{1}{\sqrt{\tau}} g\left(\frac{x}{\sqrt{\tau}}\right) = \frac{1}{\sqrt{\tau}} e^{-\pi x^2/\tau}$$

$$\Theta(\tau) = \sum_{m\in\mathbb{Z}} \hat{g}_{\sqrt{\tau}}(m) = \sum_{m\in\mathbb{Z}} \frac{1}{\sqrt{\tau}} e^{-\pi m^2/\tau} = \frac{1}{\sqrt{\tau}}\Theta(1/\tau)$$

<u>Ex.</u>: Ω: π



π

$f_{m,n}(x,y) = \sin(mx)\sin(nx)$

$Z(f_{m,n}) =$

However, If $A \neq B$ then the eigen space have $\dim > 1$.

eigenvalue $E_{m,n} = \left(\frac{\pi m}{A}\right)^2 + \left(\frac{\pi n}{B}\right)^2 = m^2 + n^2$
$A = B = \pi$

$$\dim \text{eigenspace} = \#\{(m,n)\in\mathbb{N}^2 \mid E = m^2+n^2\}$$

If $E = p \equiv 1 \pmod 4$ prime then $r_2(p) = 8$.

Diameter of p eigenspace $= 2$

$r_2(65) = 16$ so $\dim(65) = 4$ .. So we get complicated

eigenfns. $f_E(x,y) = \sum_{\substack{m^2+n^2=E \\ m,n\geq 1}} a_{m,n}\sin(mx)\sin(ny)$

<u>General theory</u>:

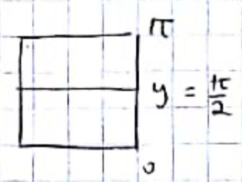
$$c_1\sqrt{E} \leq \text{length}(Z(f_E)) \leq c_2\sqrt{E}$$

$\exists c_1(\Omega) < c_2(\Omega)$ s.t $\forall$ eigenfunction $-\Delta f = E f$). $Z(f)$ is smooth outside of finite many pts. & at singular points it looks like equiangular intersecting.

Persistent components:

Does the nodal set vary with $E \to \infty$?

Example: the line $y = \frac{\pi}{2}$ is on the nodal set S.



$f_{m,n}(x,y) = \sin(mx)\sin(2ny)$

$\forall m,n \geq 1, \quad E = m^2 + (2n)^2$

Def.:

Let $\Sigma \subseteq \Omega_{\pi,\pi} = \square$ be a fixed curve. We say Σ is persistant if $\exists E_n \to \infty$ & eigenfunctions f_n with eigenvalues E_n s.t $f_n|_{\Sigma} \equiv 0 \quad \forall n$.

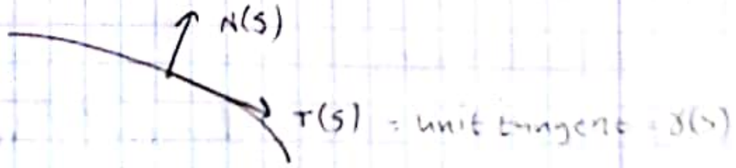
Example: $\Sigma = \{y = \frac{\pi}{2}\}$

This is essentially the only example!

Thm.:

Assume $\Sigma \subset \square$ is a real-analytic curve with nowhere vanishing curvature. Then Σ is not persistant. i.e. $\exists E_{\Sigma} > 0$ s.t if $E \geq E_{\Sigma}$ & f_E is an eigenfunction with eigenvalue E then $f_E|_{\Sigma} \not\equiv 0$.

For $\gamma : [0, L] \to \Sigma$ arclength parameterization so $\|\dot{\gamma}(s)\| = 1$



Def 1:

Curvature $K(s)$ of the pt. $\gamma(s)$ is $\dot{T}(s) = K(s)N(s)$.

Def 2:

Let $\alpha(s)$ be the angle between $T(s)$ & the x axis. Then

$$K(s) := \left|\frac{d\alpha}{ds}\right|$$

F-S eqs.:

$$\dot{T} = \kappa N \ , \quad \dot{N} = -\kappa T$$

We assume Σ has nowhere zero curvature:

$\kappa(s) \neq 0 \ \forall s \in [0,1]$. $\quad 0 < \kappa_{min} \leq \kappa(s) \leq \kappa_{max}$

Lemma:

Let $\Sigma \subseteq \mathbb{R}^2$ be a real analytic curve with nowhere vanishing curvature. Let $\gamma : [0, L] \longrightarrow \Sigma$ be an arc length parameterization. $I_\Sigma(\xi) = \int_0^L e^{i\langle \xi, \gamma(t)\rangle} dt$. Then

$$\exists C_\xi > 0 \ \text{ s.t } \ |I_\Sigma(\xi)| \leq C_\xi \frac{1}{|\xi|^{1/2}}$$

Example: Σ = unit circle, $\gamma(s) = (\cos(s), \sin(s))$

$$I_{S^1}(\xi) = J_0(|\xi|) = \int_{-\pi}^{\pi} e^{i|\xi|\cos(t)} dt$$

Proof of theorem: $\quad \vec{x} = (x, y)$

$$-\Delta f = Ef \qquad f = \sum_{\substack{m^2+n^2=E \\ m,n\geq 0}} a_{m,n} \sin(mx)\sin(ny) = \sum_{\substack{|\xi|^2 = E \\ \xi \in \mathbb{Z}^2}} a(\xi) e^{i\langle \xi, \vec{x}\rangle}$$

choose $\xi_0 = (m_0, n_0)$ s.t $|a(\xi_0)| \geq |a(\xi)| \quad \xi \in \mathbb{Z}^2 \ \forall \xi$.

After dividing by $a(\xi_0)$ we have

$$f = e^{i\langle \xi_0, x\rangle} + \sum_{\substack{|\xi|^2 = E \\ \xi \neq \xi_0}} a(\xi) e^{i\langle \xi, x\rangle} \qquad \text{notice } |a(\xi)| \leq 1$$

$$\text{Let } J = \int_\Sigma e_{-\xi_0} \cdot f = \int_0^L e^{-i\langle \xi_0, \gamma(t)\rangle} f(\gamma(t))\, dt$$

$f|_\Sigma = 0 \implies J = 0$. On the other hand

$$J = \int_0^L e^{-i\langle \xi, \gamma(t)\rangle} \left(e^{i\langle \xi_0, \gamma(t)\rangle} + \sum_{\xi \neq \xi_0} a(\xi) e^{i\langle \xi, \gamma(t)\rangle} \right) dt$$

$$= L + \sum_{\substack{\xi \neq \xi_0 \\ |\xi| = \sqrt{E}}} a(\xi) \underbrace{\int_0^L e^{i\langle \xi - \xi_0, \gamma(t)\rangle} dt}_{I(\xi - \xi_0)}$$

By vdc. lemma $I(\xi-\xi_0) \ll \frac{1}{|\xi-\xi_0|^{1/2}}$.

By Jarník's Thm, there is at most one $\xi_1 \neq \xi_0$ s.t $|\xi_1-\xi_0| < (\sqrt{E})^{1/3} = E^{1/6}$.

The sum over $\xi \neq \xi_0, \xi_1$ is bounded by:

$$\ll \sum_{\substack{\xi \neq \xi_0, \xi_1 \\ |\xi|^2 = E}} |a(\xi)| \frac{1}{|\xi-\xi_0|^{1/2}} \ll \sum 1 \cdot \frac{1}{(\sqrt{E})^{1/3}} \le$$

$$\le \frac{\#\{|\xi|^2 = E\}}{E^{1/6}} \ll \frac{E^{\varepsilon}}{E^{1/6}} \quad \forall \varepsilon > 0$$

So we find $0 = J = 1 + a(\xi_1) \cdot I(\xi_1 - \xi_0) + O\left(\frac{1}{E^{t-\varepsilon}}\right)$

If ξ_1 does not exist we get a contradiction for $E \gg 1$.

Remains to deal with $I(\xi_1 - \xi_0)$.

Trivial bound: $|I(\xi)| \le 1$. $I(\xi) = \int_0^1 e^{i\langle \xi, \gamma(t) \rangle} dt$.

Know: $|\xi_1 - \xi_0| \ge 1$.

<u>Want to avoid</u> $|I(\xi_1 - \xi_0)| = 1$

<u>Claim:</u>

If $|v| \ge 1$ then $|I(v)| \le c < 1$ (Then we win!)

It suffices to show $|I(v)| \neq 1$ for $v \neq \vec{0}$.

Because $I(v)$ is continuous, $|I(v)| \le \frac{1}{2}$ for $|v| \ge 10$ & by compactness for $1 \le |v| \le 10$, $\max_{1 \le |v| \le 10} I(v) \neq 1$

So is $\le \frac{9}{10} \cdot 1$

Cauchy Shwarts:

$$\left|\int_0^L f(t)\,g(t)\,dt\right| \leq \sqrt{\int f^2 \int g^2}$$

& equality $\iff g = \lambda f$, $\lambda \in \mathbb{C}$.

Apply to $f(t) = 1$, $g(t) = e^{i\langle v, \gamma(t)\rangle}$ So

$$\int_0^L |f|^2 = \int_0^L |g|^2 = L$$

So $|I| = L \iff e^{i\langle v, \gamma(t)\rangle} \equiv \lambda$ constant $\iff$

$\iff 0 = i\langle v, \gamma(t)\rangle e^{i\langle v, \gamma(t)\rangle}$

$\frac{d}{dt} = 0 \iff \langle v, \dot\gamma(t)\rangle \equiv 0 \ \forall t$

So $\dot\gamma(t) \equiv u$ is fixed $\Rightarrow \gamma(t) = tu + u_0$ is a straight line