Lecture 1: 7/3/2017

Topics: "open problems in number Theory"

1) The golden Ratio. "normal"

$$\underset{\text{irrational}}{\varphi} = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}} = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}, \quad F_{n+1} = F_n + F_{n-1} \text{ Fibonacci sequence.}$$

Def:

A number $\alpha = \dots . a_1 a_2 \dots a_n \dots$ is simply normal (in base 10) if every digit $(0,1,2,\dots,9)$ appears with same frequency $= 1/10$

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \leq N \mid a_n = 7\} = 1/10, \text{ ditto for any digit } t = 0,1,2,\dots,9.$$

Def:

A real number $\alpha = \dots . a_1 a_2, \dots, a_n \dots$ is Normal (to base 10) if $\forall k \geq 1$, every string of k digits (eg $k=3$, 127) appears with the same frequency $= 1/10^k$

ex:

Find a simply normal number which is NOT normal (to base 2)

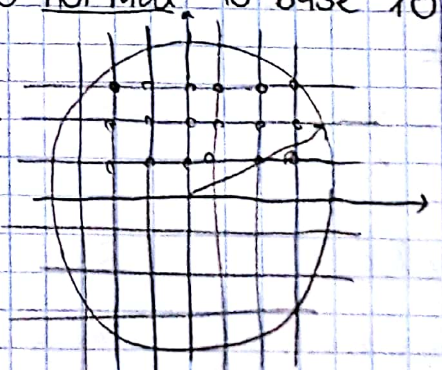
Question: real algebraic, π, e, ...

Is any "naturally occurring" number Normal (to base 10)

non-natural example (Champernowne 1932)

0.123456789101112131415... is normal to base 10.

2) "The circle problem" (Hardy 1920's).



count lattice pts in a ball

$$N(R) = \#\{n = (n_1, n_2) \in \mathbb{Z}^2 \mid n_1^2 + n_2^2 \leq R^2\}$$

Easy: $N(R) \sim \text{area } B(0,R) = \pi R^2$

$$P(R) := N(R) - \pi R^2$$

$P(R) = o(R^2)$ $\frac{P(R^2)}{R^2} \longrightarrow 0$

little oh

<u>Gauss</u>: $P(R) \ll R \simeq$ length of circumference

<u>Hardy's conjecture</u>:

$$P(R) \ll \left(\sqrt{\text{length circumference}}\right)^{1+\varepsilon} \quad \forall \varepsilon>0, \quad \ll R^{1/2+\varepsilon}$$

Sierpinski Van der Corput 1906: $P(R) \ll R^{2/3}$

3) Let $r(n) = \#$ of lattice pts <u>on</u> the circle $x^2+y^2=n$

So $$\sum_{n \le R^2}{}' r(n) = N(R) \sim \pi R^2$$

So "on average" over all $n \le R^2$ $r(n)$ is π. But infact for "almost all" n's, $r(n)=0$.

<u>E. Landau (1908)</u>:

$$\#\{ n \le N \mid n = \square + \square \} \sim K \frac{N}{\sqrt{\log N}}$$

$\Longrightarrow$ The "average gap" between consecutive integers $n = \square + \square \approx N$ is $\asymp \sqrt{\log N}$

<u>Question</u>:

What is the maximal gap?

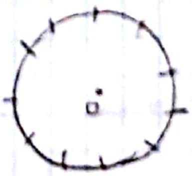
Let order: $n_1 = 1 = 1^2+0^2$, $n_2 = 2 = 1^2+1^2$, $n_3 = 4 = 2^2+0^2$, $n_4 = 5 = 2^2+1^2 \ldots$

$n_i = i^{th}$ sum of 2 squares

<u>Conj.</u>:

$$G(x) \max_{n_i \le x} (n_{i+1} - n_i) \ll x^{\varepsilon}, \quad \forall \varepsilon > 0$$

<u>Folklore</u>: $G(x) \le x^{1/4}$



4) Lattice points in short arcs.

Take a circle of radius $R = \sqrt{n}$, have $r(n)>0$ lattice pts on it.

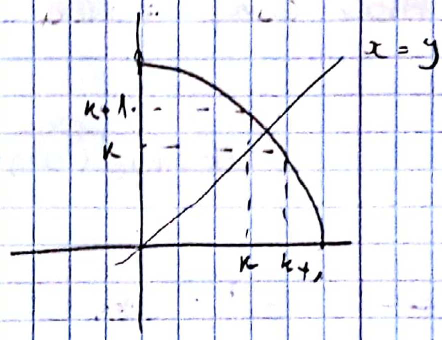
Can we have <u>close</u> lattice pts? Total number is $r(n)$.

length of circle is $2\pi\sqrt{n} = 2\pi R$

$\Longrightarrow$ average distance is $\frac{2\pi\sqrt{n}}{r(n)} \gg n^{1/2 - \varepsilon}$

<u>Know:</u>

$$r(n) \ll n^{\varepsilon} \quad \forall \varepsilon > 0.$$



<u>Example:</u>

$$n = k^2 + (k+1)^2, \quad A = (k, k+1), \quad B = (k+1, k)$$

$$\text{dist}(A, B) = \sqrt{2} = O(1)$$

<u>Jarnik (1940 ?):</u>

An arc of length $< R^{1/3}$ cannot contain 3 lattice points.

<u>Conj.:</u>

Fix $\delta < 1$. Then an arc of length R^{δ} contains at most $M(\delta) < \infty$ lattice points.

<u>Jarnik:</u>

ok $\delta = 1/3$, $M(1/3) = 3$

<u>Cilleruelo & Cordoba (1992):</u>

ok $\delta < \frac{1}{2}$

$$\boxed{?? \quad \delta = 1/2 \quad ??}$$

5) The ABC conjecture (Masser & Oesterlé 1985)

A, B, C coprime integers s.t. $A + B = C$

$\forall \varepsilon > 0 \, \exists c_\varepsilon > 0. \max(|A|, |B|, |C|) < C_3 \operatorname{rad}(ABC)^{1+\varepsilon}$

Where: $\operatorname{rad}\left(\prod p_j^{l_j}\right) = \prod p_j$

$\Longrightarrow$ An asymptotic version of Fermat's last theorem Wiles (1994)

($n \geq 3$ $x^n + y^n = z^n$ only has trivial solutions)

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Pf (ABC $\Rightarrow$ Fermat):

$A = x^n$, $B = y^n$, $C = z^n$, x, y, z coprime.

ABC conj $\Rightarrow \forall \varepsilon > 0.\ C = z^n < C(\varepsilon)\, \mathrm{rad}(x^n y^n z^n)^{1+\varepsilon}$

$\mathrm{rad}(x^n y^n z^n) = \mathrm{rad}(xyz)$

$$= C(\varepsilon)\, \mathrm{rad}(xyz)^{1+\varepsilon} \underset{\mathrm{Rad}(N) \le N}{\le} C(\varepsilon)(xyz)^{1+\varepsilon} \le C(\varepsilon) z^{3+3\varepsilon}$$

$\Rightarrow z^n < C(\varepsilon) z^{3+\varepsilon} \leftrightarrow z^{n-3-\varepsilon} < C(\varepsilon)$

$\varepsilon = \frac{1}{2} \Rightarrow \exists\, C(\frac{1}{2}) = c > 0$ s.t $\forall x, y, z$, $\forall n \ge 4$: $z^{1/2} \le z^{n-3-\frac{1}{2}} < c$

$\Rightarrow z < c^2$

$\Rightarrow \forall n \ge 4.\ \exists$ only finitely many solutions to $x^n + y^n = z^n$

(Proved by Faltings 1984)

Normal Numbers

Def:

$\alpha = 0.a_1 a_2 \dots \in (0,1)$ is normal to base 10 if $\forall k \ge 1$, $\forall$ string $t_1 \dots t_k$, $t_i \in \{0, \dots, 9\}$

$$\frac{1}{N} \# \{ n \le N \mid a_{n+1} = t_1, a_{n+2} = t_2, \dots, a_{n+k} = t_k \} \xrightarrow[N \to \infty]{} \frac{1}{10^k}$$

Ex: $\frac{1}{2}$ is not normal $\quad \frac{1}{2} = 0.500\dots = 0.4999\dots$

Exercise:

Rationals are never normal.

Un-Natural Examples:

Champernowne 1952: $0.123456789101112\dots$ normal to base 10.

Copeland & Erdos 1946, Conj. by Champernowne:

$0.235711131719 23\dots$ normal to base 10.

Davenport & Erdos 1952 (Conj. Copeland & Erdos):

take $f(x) \in \mathbb{Z}[X]$ s.t $f(x) > 0$, $x = 1, 2, 3, \dots$, take $\alpha = 0.f(1)f(2)f(3)\dots$ is normal to base 10.

Conj. (E. Borel 1950):

real algebraic $\alpha \notin \mathbb{Q}$ is normal to any base.

Problem:

Find a "natural" example of a normal number.

Conj: π, e normal.

Uniform Distribution

Let $X = \{x_1, x_2, x_3, \dots\} \subset \mathbb{R}/\mathbb{Z} \simeq [0,1)$ be a sequence of points on the circle.

We say X is uniformly distributed if $\forall$ interval $I = [a,b] \subset \mathbb{R}/\mathbb{Z}$

$$\lim_{N \to \infty} \frac{1}{N} \# \{ n \le N \mid x_n \in I \} = \text{length } I$$

Note:

X is U.D $\Longrightarrow$ X is dense in $\mathbb{R}/\mathbb{Z}$

Generally, $X = \{x_n\} \subset \mathbb{R}$ then look at fractional parts

$x = \text{integer} + \{x\}$, $0 \le \{x\} \le 1$. then X is U.D mod 1 if the fractional parts are U.D.

Ex:

$x_n = \{\log n\}$, $n = 1, 2, 3, \dots$ is dense in $[0,1)$. we
fractional part.
will see that it is not uniformly distributed.

Example: (Soon)
fractional part
$\alpha \in \mathbb{R} \setminus \mathbb{Q}$ irrational, $x_n = \{\alpha \cdot n\}$, $n = 1, 2, 3, \dots$

Claim:

This is uniformly distributed $\Longrightarrow$ dense.

Proposition:

α normal to base 10 $\iff$ $\{\alpha \cdot 10^n\}$ is U.D.

Proof:

Assume α is normal. want U.D. Take interval $I \subset [0,1)$ & show $\frac{1}{N}\{n \le N \mid \{\alpha \cdot 10^n\} \in I\} \xrightarrow[N \to \infty]{} \text{length}(I)$

check, WLOG: $I = [0.a_1a_2\ldots a_k, \ 0.a_1\ldots(a_k+1))$

$\#\{n \le N \mid \{\alpha \cdot 10^n\} \in [0.123, 0.124)\}$

$\alpha = 0.537189327$

$10\alpha = 5.3718\ldots \qquad \{10\alpha\} = 0.37189\ldots$

$\{10^2\alpha\} = 0.71893$

$\{10^3\alpha\} = 0.1893\ldots$

$\{10^n\alpha\} = 0.a_{n+1}a_{n+2}\ldots \qquad \alpha = 0.a_1a_2\ldots$

$\{10^n\alpha\} \in I = [0.123, 0.124) \iff a_{n+1} = 1, a_{n+2} = 2, a_{n+3} = 3$

$\frac{1}{N}\#\{n \le N \mid \{\alpha \cdot 10^n\} \in I\} = \frac{1}{N}\#\{n \le N \mid a_{n+1} = 1, a_{n+2} = 2, a_{n+3} = 3\}$

So UD: LHS $\longrightarrow$ length$(I) = 0.001 = \frac{1}{10^3}$

Normality: RHS $\longrightarrow \frac{1}{10^3}$ so we get equivalence

H. Wyel (1916): Weyl's criterion

$X = \{x_n \mid n = 1,2,3,\ldots\} \subseteq \mathbb{R}$ is U.D mod 1 $\iff$

$\forall$ integer $k \neq 0$ $\quad \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k x_n} \xrightarrow[n \to \infty]{} 0$

Thm.:

If $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ irrational, then $\{\alpha n \mid n = 1,2,\ldots\}$ is U.D mod 1.

Look at Weyl sums $S_k(N) = \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k \alpha n} =$

$= \frac{1}{N}\sum_{n=1}^{N}(e^{2\pi i k\alpha})^n =$

$$= \begin{cases} 1, & e^{2\pi i k\alpha} = 1 \text{ cannot happen if } \alpha \notin \mathbb{Q} \\ \frac{1}{N}\frac{(e^{2\pi i k\alpha}) - (e^{2\pi i k\alpha})^{N+1}}{1 - e^{2\pi i k\alpha}} & \end{cases}$$

$$|S_k(N)| \leq \frac{1}{N}\frac{2}{|1-e^{2\pi i k\alpha}|} \xrightarrow[\alpha\to\infty]{} 0$$

Example: That can be done using Weyl's sums: $\alpha n^2, \alpha \notin \mathbb{Q}$

Example:

The sequence of fractional parts of $\log n$ in not unif. distributed

By Weyl's criterion, it suffices to show:

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i \log n} \not\xrightarrow[\alpha\to\infty]{} 0$$

Summation By Parts: (Abel's Identity)

Let $\{a(n)\}_{n=1}^{\infty}$ be a sequence, $f(t)$ diffrentiable.

$$A(x) := \sum_{1\leq n\leq x} a(n) \qquad (A(x) := 0,\ x<1)$$

Then $\displaystyle\sum_{1\leq n\leq x} a(n)f(n) = A(x)f(x) - \int_1^x A(t)f'(t)\,dt$

To compare: Integration by parts:

$$\int_1^x A'(t)f(t)\,dt = A(t)f(t)\Big|_1^x - \int_1^x A(t)f'(t)\,dt$$

Apply to $f(t) = e^{2\pi i \log t}$, $a(n) = 1$.

$$\sum_{n=1}^{N} e^{2\pi i \log n} = \sum_{1\leq n\leq N} a(n)f(n) \ (=)$$

$f(t) = t^{2\pi i}$, $f'(t) = 2\pi i\, t^{2\pi i - 1}$, $A(t) = \sum_{1\leq n\leq t} a(n) = \sum_{1\leq n\leq t} 1 = \lfloor t\rfloor$

$$(=)\ A(N)f(N) - \int_1^N A(t)f'(t)\,dt = N N^{2\pi i} - \int_1^N \lfloor t\rfloor\, 2\pi i\, t^{2\pi i - 1}\,dt = \quad t = \lfloor t\rfloor + \{t\}$$

$$= N N^{2\pi i} - 2\pi i\int_1^N t\, t^{2\pi i - 1}\,dt + 2\pi i \int_1^N \{t\}\, t^{2\pi i - 1}\,dt = \quad \{t\} \leq 1,\ |t^{2\pi i}| = 1$$

$$= N N^{2\pi i} - 2\pi i \left.\frac{t^{2\pi i + 1}}{2\pi i + 1}\right|_1^N + O\left(\int_1^N \frac{1}{t}\,dt\right) =$$

$$= N N^{2\pi i} - 2\pi i\frac{N^{1+2\pi i}}{1+2\pi i} + O(1) + O(\log N)$$

$$\Longrightarrow \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i \log n} = \frac{N^{2\pi i}}{1+2\pi i} + \underbrace{O\left(\frac{\log N}{N}\right)}_{\to 0} \not\xrightarrow[N\to\infty]{} 0$$

Proof of Summation by Parts: (assume x · integer)

$$\sum_{n=1}^{x} a(n)f(n) = \sum_{n=1}^{x}\{A(n)-A(n-1)\}f(n) =$$

$$a(n) = A(n) - A(n-1)$$

$$= \sum_{n=1}^{x} A(n)f(n) - \sum_{n=1}^{x} \underbrace{A(n-1)}_{m} f(\underbrace{n}_{m+1}) = \sum_{n=1}^{x} A(n)f(n) - \sum_{m=0}^{x-1} A(m)f(m+1) =$$

$$= A(x)f(x) - \overset{0}{\overset{\|}{A(0)}} f(1) - \sum_{n=1}^{x-1} A(n)\{f(n+1) - f(n)\} =$$

$$= A(x)f(x) - \sum_{n=1}^{x-1} A(n) \int_{n}^{n+1} f'(t)\,dt =$$

$$= A(x)f(x) - \sum_{n=1}^{x-1} \int_{n}^{n+1} \underset{\overset{\|}{A(t)}}{A(n)} f'(t)\,dt = A(x)f(x) - \sum_{n=1}^{x-1} \int_{n}^{n+1} A(t)f'(t)\,dt =$$

$$= A(x)f(x) - \int_{1}^{x} A(t)f'(t)\,dt.$$